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Manual for the estimation of first and second order regression parameters

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PREFACE

Regression analysis is an important part of today's decision making, and many of our decisions are based on projections or estimates of future unknown events. Many books on regression have been published recently. Some of them are very theoretical and focus on only a few selected topics of regression analysis. This manual is about numerical procedure to evaluate deterministic models that can be used to produce estimates or short term projections. Our objective is to provide an intermediate level discussion of statistical regression methods and to bridge the gap between theory and practise.

Some knowledge of the theory of statistics is assumed. I have written primarily for agricultural researchers, but I hope also that scientists and technologists interested in applying statistical regression methods will, by concentrating on the examples, find something useful here.

I am grateful to Jose Arze M.S. at CATIE in Turrialba, Costa Rica and Ricardo Escobar M.S. with IICA Brazil for extremely helpful comments. I acknowledge also the help of Ms. Lenora Boyce who typed the computer programmes and made test runs on the Wang Computer at the Ministry of Agriculture and Natural Resources (MANR) Barbados. I appreciate very much the patient and careful typing of Mrs. Judith Cobham at Barbados' IICA Office.

I could not possibly discuss every issue in statistical regression. However, I hope this manual provides the background that will allow the agricultural researchers to adopt the procedures included in this manual to their particular needs.

Victor Quiroga

BRIDGETOWN, BARBADOS
March, 1986

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REGRESSION

Regression is a statistical procedure used when one wants to study the relationship between variables. For example, you might hypothesize that quantity demanded of a commodity in a market is a function of its price, or that population increases as time passes or consumer expenses are taken as a function of income and so forth.

Linear Regression

Let's assume we are dealing with a single relation and that it contains only two variables, denoting them by X_i and Y_i. We may postulate:

$$Y = f(x)$$

This function identifies the variable X which is thought to influence the other variable Y. The simple relationship between these variables are expressed through the equation of a straight line:

$$Y = \hat{a} + \hat{b} X \qquad [1]$$

Straight line defined in [1] intercepts the Y axis at $\hat{a}$, so $\hat{a}$ is called the intercept. The coefficient $\hat{b}$, is the slope of the straight line; it represents the change in Y for each change in X. Also [1] implies a set of Y values called dependent Y variable and a set of X values called independent X variable. Both sets will define a sample of observation as follows

Variable X	Variable Y
X_1	Y_1
X_2	Y_2
X_n	Y_n

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When you plot Y against X you get a scatter diagram (Fig.1)

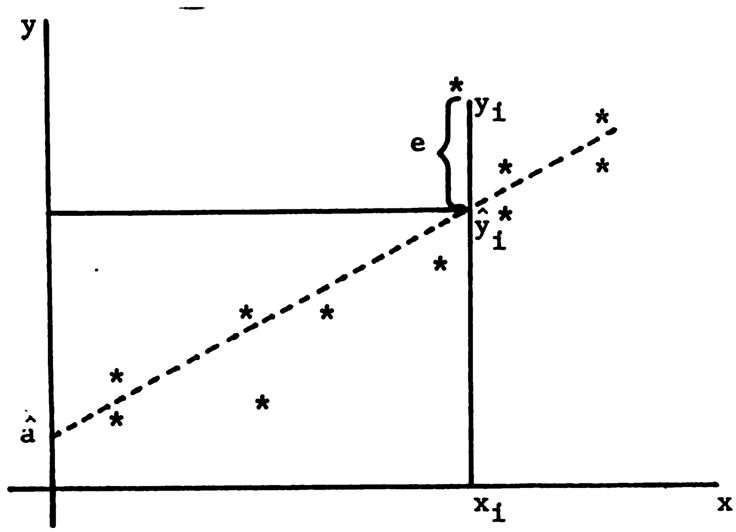

Fig 1. Regression of Y on X

Also the dotted line is an estimate line of the "Trueline", and denotes this estimate regression line by:

$$\hat{Y} = \hat{a} + \hat{b} X$$

or

$$\hat{Y}_i = \hat{a} + \hat{b} X_i \qquad [2]$$

Where $\hat{a}$ and $\hat{b}$ are merely estimates of two unknown parameters and $\hat{Y}_i$ is the ordinate on regression line for a given value X_i

To fit such estimate lines we must develop mathematical formulae for $\hat{a}$ and $\hat{b}$ based on the sample observation.

Let us add using [2] a particular Y_i on Fig 1. as well as its corresponding $\hat{Y}_i$ such as:

$$e = Y - \hat{Y}$$

$$e_i = Y_i - \hat{Y}_i \qquad [3]$$

The distance from a point Y_i to the regression line is called residual (the difference between the actual Y_i value and the $\hat{Y}_i$ value that the regression line projects).

Principle of Least Squares

These differences or residuals will be negative or positive depending on whether the actual point Y_i lies below or above the regression line. If these residuals are squared and summed, the resultant quantity must be nonnegative and will be directly proportional to the spread of the points from the regression line [4]. In other words the sum of squares of the residuals is a function of the estimated regression parameters $\hat{a}$ and $\hat{b}$ such:

$$Se_i^2 = f(\hat{a},\hat{b}) \qquad [4]$$

A necessary condition to make Se^2_i as small as possible is that the partial derivatives with respect to $\hat{a}$ and $\hat{b}$ should be zero. If we sum the squares in [3] we will have:

$$Se_i^2 = (Y_i - \hat{Y}_i)^2$$

If we replace $\hat{Y}_i$ with its value defined in 2 we get:

$$Se^2_i = S\,(Y_i - \hat{a} - \hat{b}\,X_i)^2$$

so that

$$\frac{\partial(Se^2_i)}{\partial\hat{a}} = -\,2S\,(Y_i - \hat{a} - \hat{b}\,X_i) = 0 \qquad [5]$$

and

$$\frac{\partial(Se^2_i)}{\partial\hat{b}} = -\,2SX_i\,(Y_i - \hat{a} - \hat{b}X_i) = 0 \qquad [6]$$

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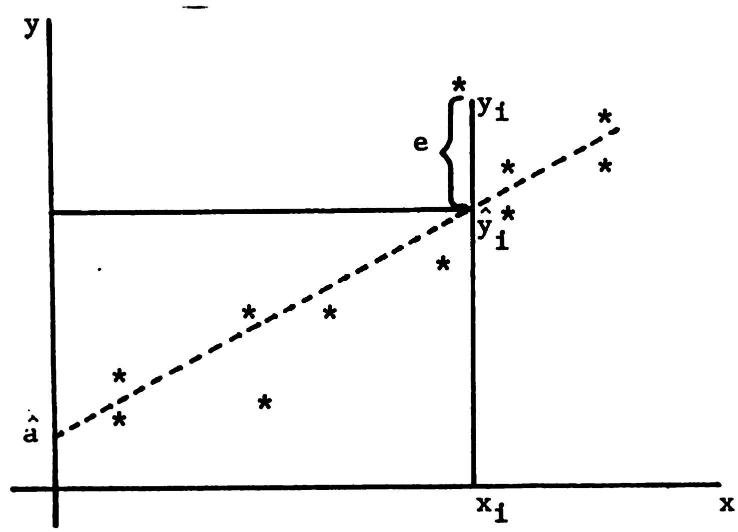

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$$\frac{\partial(Se^2_i)}{\partial\hat{b}} = -\,2SX_i\,(Y_i - \hat{a} - \hat{b}X_i) = 0 \qquad [6]$$

Simplifying equation [6] gives:

$$SX_iY_i - \hat{a}SX_i - \hat{b}\,SX_i^2 = 0$$

$$SX_iY_i = \hat{a}SX_i + \hat{b}SX_i^2$$

Using transformed values defined by $X_i = (X_i - \bar{X})$ we have:

$$S(X_i - \bar{X})\,(Y_i - \bar{Y}) = \hat{a}S\,(X_i - \bar{X}) + \hat{b}S(X_i - \bar{X})^2$$

Also by definition $S(X_i - \bar{X}) = 0$ such as:

$$S(X_i - \bar{X})\,(Y_i - \bar{Y}) = \hat{b}S(X_i - \bar{X})^2$$

$$\hat{b} = \frac{S(X_i - \bar{X})(Y_i - \bar{Y})}{S(X_i - \bar{X})^2} \qquad [7]$$

Simplifying equation [5] we get:

$$SY_i - S\hat{a} - \hat{b}SX_i = 0$$

$$SY_i = S\hat{a} + \hat{b}SX_i$$

$$SY_i = n\hat{a} + \hat{b}SX_i \qquad [8]$$

Dividing each side of equation [8] by n we get:

$$\frac{SYi}{n} = \frac{n\hat{a}}{n} + \frac{\hat{b}SXi}{n}$$

$$\bar{Y} = \hat{a} + \hat{b}\,\bar{X}$$

$$\hat{a} = \bar{Y} - \hat{b}\,\bar{X} \qquad [9]$$

So equations [9] and [7] are solutions sought.

Measures of Estimator's Variability

If $\hat{b}$ is estimated from a sample point, the value of $\hat{b}$ will vary from sample to sample. We know, however, from statistical theory that in the long run the mean of $\hat{b}$'s will coincide with the population value b and we can estimate the variance of the sampling variability of $\hat{b}$;

Rewrite equation [7] as follows:

$$\hat{b} = \frac{S(X_i - \bar{X})Y_i}{S(X_i - \bar{X})^2}$$

$$\hat{b} = SW_iY_i \qquad [10]$$

$$\text{Since } W_i = \frac{(X_i - \bar{X})}{S(X_i - \bar{X})^2}$$

Let us evaluate variances in both sides of equation [10]:

$$VAR\ (\hat{b}) = VAR\ (SW_iY_i)$$

$$= SW_i^2 * VAR\ (Y_i)$$

$$= \frac{1}{S(X_i - \bar{X})^2} * R.M.S$$

$$VAR\ (\hat{b}) = \frac{\text{Residual M.S.}}{S(X_i - \bar{X})^2} \qquad [11]$$

Let us evaluate variances in both sides of [9] :

$$VAR(\hat{a}) = VAR(\bar{Y} - \hat{b}\bar{X})$$

$$= VAR(\bar{Y}) + \bar{X}^2 * VAR(\hat{b})$$

$$= \frac{R.M.S}{n} + \frac{\bar{X}^2 * R M S}{S(X_i-\bar{X})^2}$$

$$VAR(\hat{a}) = \text{Residual M.S.} \left\{ \frac{1}{n} + \frac{\bar{X}^2}{S(X_i-\bar{X})^2} \right\} \quad [12]$$

Equations [11] and [12] are solutions sought.

Test of Hypothesis

Now we may do a test of hypothesis on $\hat{b}$ as follows:

$$H_0 : B = 0$$

$$H_A : B \neq 0$$

$$t = \frac{\hat{b} - B}{VAR(\hat{b})}$$

With (n-2) degrees of freedom, if the computed t value is larger than the statistical table's critical value for a given level of significance, say 0.05, the null hypothesis that B = 0 would be rejected. Otherwise it would be concluded that the observed $\hat{b}$ is not significant at the 0.05 level.

Analysis of Variance

Total variation is split in two components. One accounts for the variation due to regression and the other accounts for the residual variation. From [3] we derive formulae for sum of squares due to regression, residual and total.

$$e_i = Y_i - \hat{Y}_i$$

$$= Y_i - \hat{Y}_i + \bar{Y} - \bar{Y}$$

$$Se^2_i = S(Y_i - \hat{Y}_i + \bar{Y} - \bar{Y})^2$$

$$= S\left\{(Y_i - \bar{Y}) - (\hat{Y}_i - \bar{Y})\right\}^2$$

$$= S\,(Y_i - \bar{Y})^2 + S(\hat{Y}_i - \bar{Y})^2 - 2S(Y_i - \bar{Y})(\hat{Y}_i - \bar{Y})$$

$$= S(Y_i - \bar{Y})^2 + S(\hat{Y}_i - \bar{Y})^2 - 2S(\hat{Y}_i - \bar{Y})^2$$

$$= S(Y_i - \bar{Y})^2 - S(\hat{Y}_i - \bar{Y})^2$$

$$S(Y_i - \hat{Y}_i)^2 = S(Y_i - \bar{Y})^2 - S(\hat{Y}_i - \bar{Y})^2$$

$$S(Y_i - \bar{Y})^2 = S(\hat{Y}_i - \bar{Y})^2 + S(Y_i - \hat{Y}_i)^2$$

Total S.S. = Regression S.S. + Residual S.S.

Analysis of variance is presented in tabular format as follows:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Fisher's F
Regression	$S(\hat{Y}_i - \bar{Y})^2$	1	RSS/DF	RMS/rMS
Residual	$S(Y_i - \hat{Y})^2$	n-2	rSS/DF	
TOTAL	$S(Y_i - \bar{Y})^2$	n-1		

Measure of Reliability

A natural measure of prediction accuracy and the strength of linear relationship is the ratio of explained variation in the dependent variable Y_i to the total variation in Y_i: This ratio is sometimes referred as the coefficient of determination.

$$R2 = \frac{S\,(\hat{Y}_i - \bar{Y})^2}{S\,(Y_i - \bar{Y})^2}$$

First Order Linear and Nonlinear Models

Obvious extensions are now required to cover the case of nonlinear relationships between two variables. Economic theory may suggest that the relationship between two variables can be adequately represented only by a nonlinear form but further inspection of the scatter diagram may indicate the inappropriateness of attempting to fit linear relationship.

The most commonly nonlinear relationship are displayed in Figures 2 - 5.

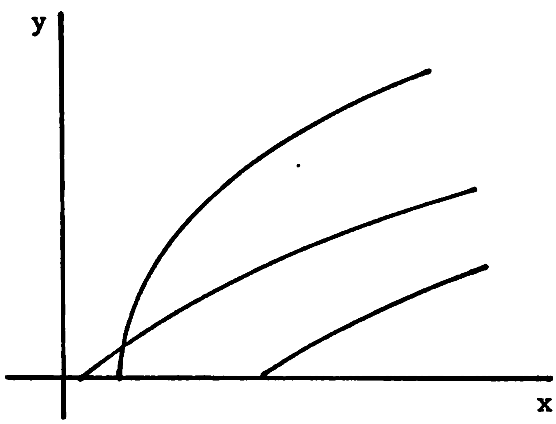

Figure 2. Semilog Regression

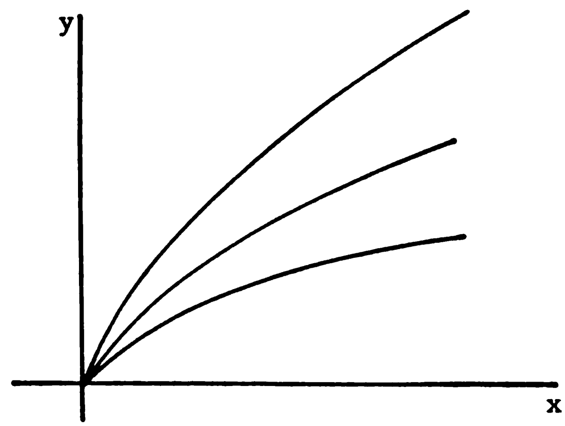

Figure 3. Logarithm Regression

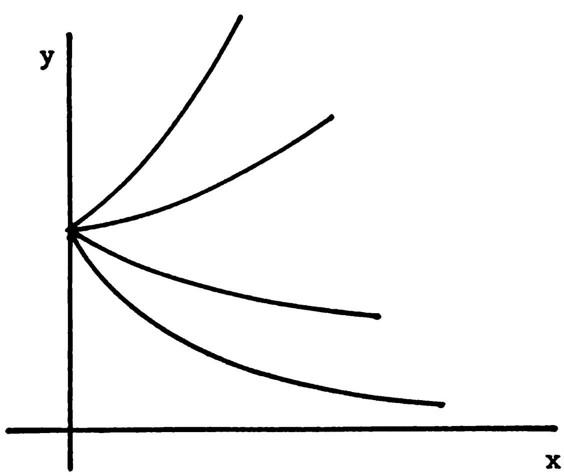

Figure 4. Geometric Regression

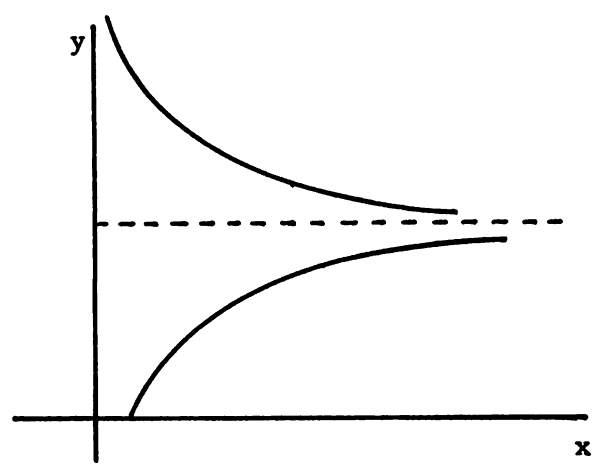

Figure 5. Reciprocal Regression

To solve for regression fitness we have to perform appropiate transformation in the original sample points and apply the procedure described early. Five examples follow to illustrate computation of First Order Models.

Analysis of variance is presented in tabular format as follows:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Fisher's F
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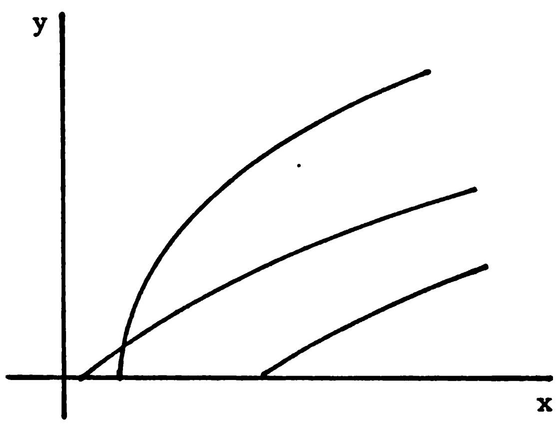

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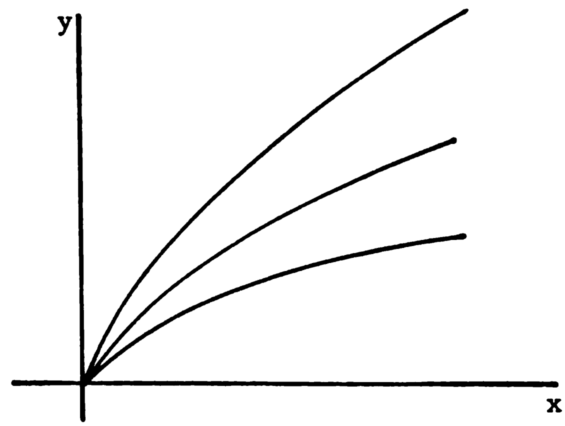

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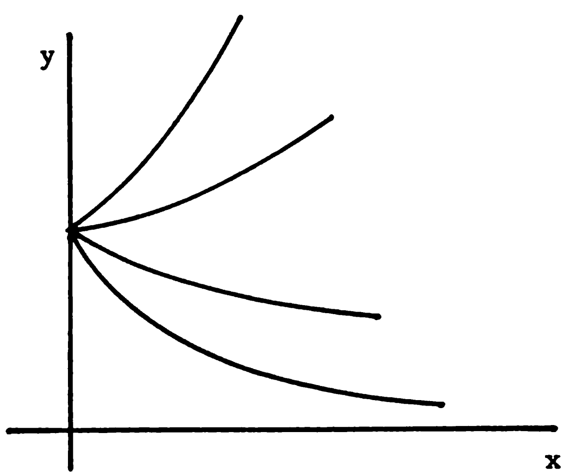

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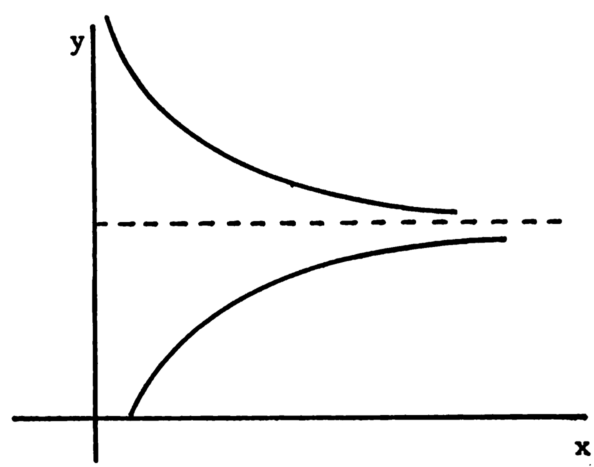

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To solve for regression fitness we have to perform appropiate transformation in the original sample points and apply the procedure described early. Five examples follow to illustrate computation of First Order Models.

Five Examples of First Order Linear Models

Linear Regression

$$Y_i = a + b\,X_i$$

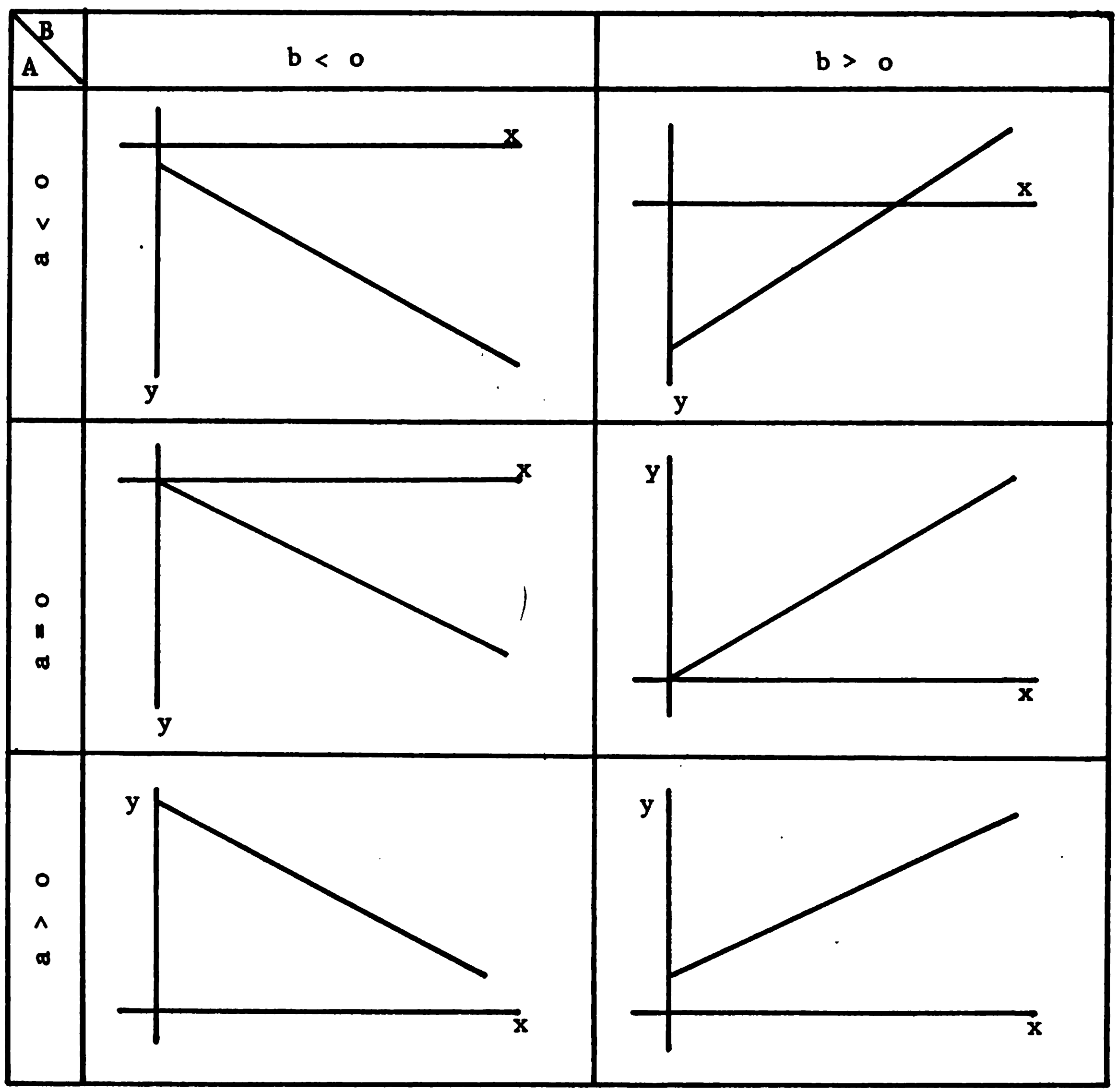

Figure 6. Spectrum of linear regression according to values of â and b̂.

EXAMPLE 1

An investigator wants to explore the possible relationship between concentration of drugs and time elapsed. A drug was administered to a patient by intravenous injection; blood samples were taken over nine (9) days and assayed for drug content as follows:

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Establish a first order linear regression for the data.

b) Estimate population parameter based on the sample.

c) Examine estimated plot, scatter diagram and the coefficient of determination R2 and decide whether the linear model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limit in tabular form.

Tabulation of Sum Squares and Cross Products

	X_i	Y_i	$X_i * Y_i$	$X_i ** 2$	$Y_i ** 2$
	1.000	8.000	8.000	1.000	64.000
	2.000	10.000	20.000	4.000	100.000
	3.000	9.000	27.000	9.000	81.000
	4.000	8.000	32.000	16.000	64.000
	5.000	7.000	35.000	25.000	49.000
	6.000	6.000	36.000	36.000	36.000
	7.000	6.000	42.000	49.000	36.000
	8.000	3.000	24.000	64.000	9.000
	9.000	2.000	18.000	81.000	4.000
SUM	45.000	59.000	242.000	285.000	443.000
MEAN	5.00	6.55555			

Computation of Estimators:

$$\hat{b} = \frac{SXY - SX*SY/n}{SX2 - SX*SX/n}$$

$$= \frac{242 - 45 * 59/9}{285 - 45 * 45/9} = \frac{-53}{60}$$

$$= - 0.883333$$

$$\hat{a} = \bar{Y} - \hat{b} * \bar{X}$$

$$= 6.55555 - (-0.883333) * 5$$

$$= 10.972215$$

Total S.S. = SY2 - SY * SY/n

= 443 - 59 * 59/9 = 56.2222

Regression S.S. = $\hat{b}$ * (SXY - SX * SY/n)

= -0.883333 * (242 - 45 * 59/9) = 46.8166

Residual S.S. = Total S.S. - Regression S.S.

= 56.2222 - 46.8166 = 9.4056

Tabulation of Linear Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	46.8166	1	46.8166	34.84
Residual	9.4056	7	1.34365	
Total	56.2222	8		

Computation of Reliability and Student's T test:

$$R\,2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{46.8166}{56.2222} * 100 = 83.27\%$$

$$sb = \sqrt{\frac{\text{Residual M.S.}}{SX2 - SX * SX/n}}$$

$$= \sqrt{\frac{1.34365}{285 - 45 * 45/9}} = 0.149646$$

$$t = \frac{\hat{b}}{Sb}$$

$$= \frac{-0.883333}{0.149646} = -5.90$$

<u>Computation of $\hat{Y}_i$ Estimates and Standard Error of $\hat{Y}_i$:</u>

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i)$$

$$\hat{Y}_1 = 10.972215 - 0.883333 * (1) = 10.089$$

$$\hat{Y}_2 = 10.972215 - 0.883333 * (2) = 9.206$$

$$S\hat{Y}_i = \sqrt{\text{Residual M.S.} * \left\{\frac{1}{n} + \frac{(X_i - \bar{X})^2}{S(X_i - \bar{X})^2}\right\}}$$

$$S\hat{Y}_1 = \sqrt{1.34365 * \left\{\frac{1}{9} + \frac{(1 - 5)^2}{60}\right\}}$$

$$= \sqrt{1.34365 * 0.11111 + 0.26666} = 0.7125$$

$$S\hat{Y}_2 = \sqrt{1.34365 * 0.11111 + 0.15} = 0.5923$$

$$E_i = t * S\hat{Y}_i; \text{ for 0.05 probability level}$$

$$E_1 = 2.365 * 0.7125 = 1.685$$

$$E_2 = 2.365 * 0.5923 = 1.401$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 10.089 - 1.685 = 8.404$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 10.089 + 1.685 = 11.774$$

Same computation applies to $\hat{Y}_2$, $\hat{Y}_3$, $\hat{Y}_4$ and so forth.

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
1.000	8.000	10.089	1.685	8.404	11.774
2.000	10.000	9.206	1.401	7.805	10.606
3.000	9.000	8.322	1.156	7.166	9.478
4.000	8.000	7.439	0.980	6.459	8.419
5.000	7.000	6.556	0.914	5.642	7.469
6.000	6.000	5.672	0.980	4.692	6.652
7.000	6.000	4.789	1.156	3.633	5.945
8.000	3.000	3.906	1.401	2.505	5.306
9.000	2.000	3.022	1.685	1.337	4.707

Scatter Diagram and Plot of Linear Regression Functions:

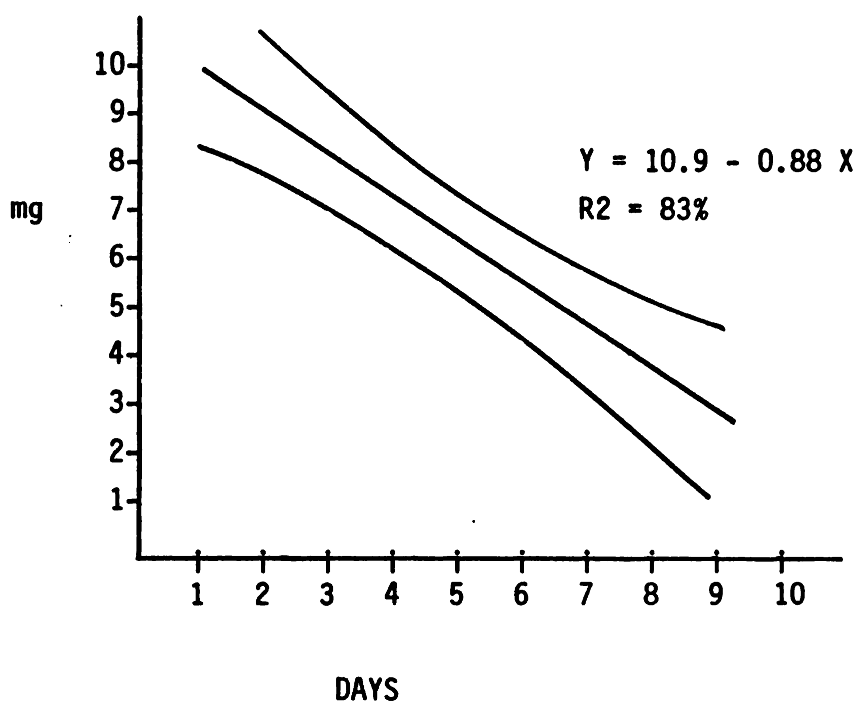

Figure 7. Linear Regression of concentration on days.

Semilogarithm Regression

$$Y_i = a + b \log X_i$$

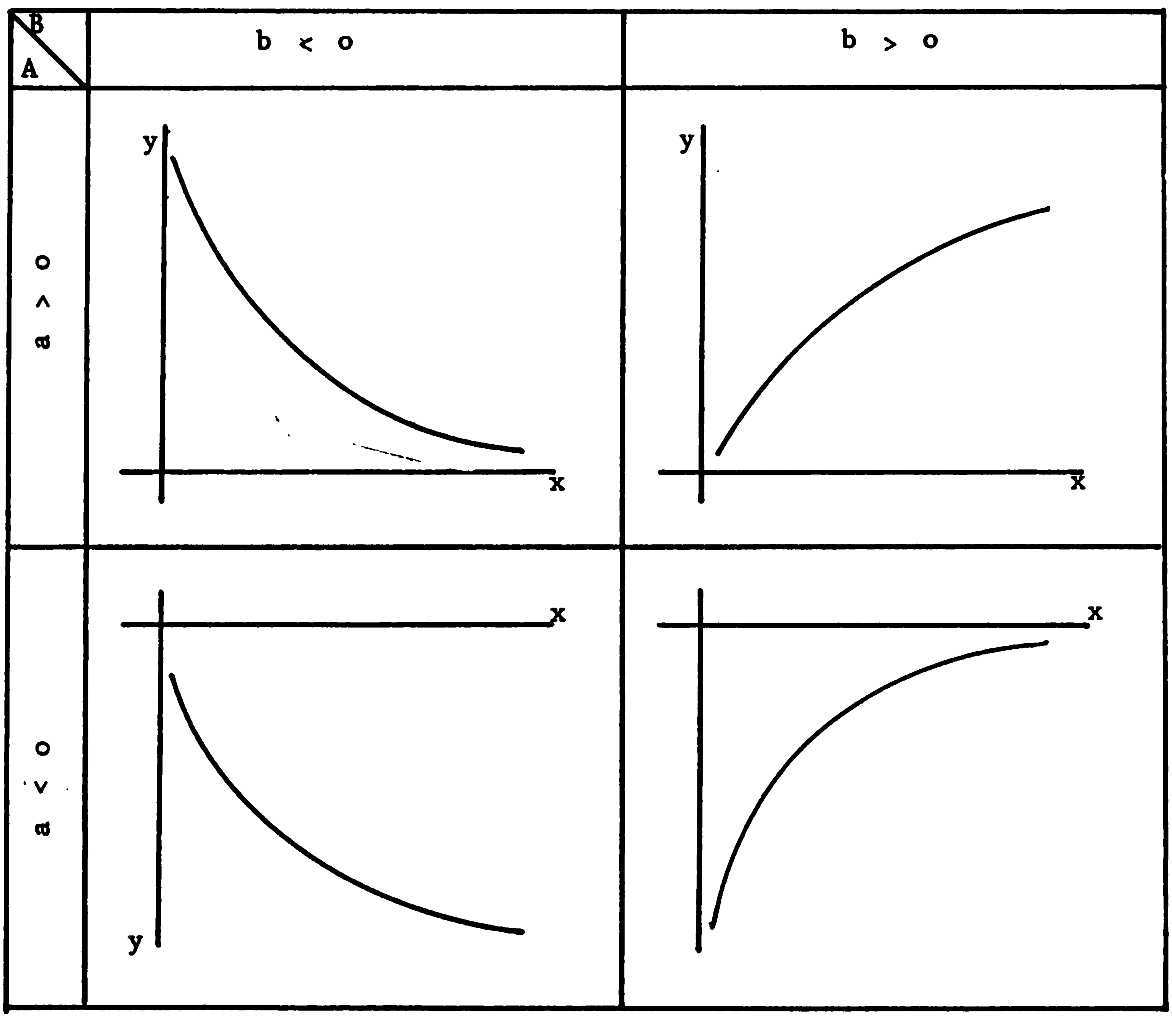

Figure 8. Spectrum of Semilogarithm Regression according to value of $\hat{a}$ and $\hat{b}$

EXAMPLE 2.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Establish a first order nonlinear regression for sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the Semilogarithm model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limit in tabular form.

Tabulation of Sum Squares and Cross Products

	X_i	Y_i	$X_i * Y_i$	$X_i ** 2$	$Y_i ** 2$
	0.0000	8.000	0.0000	0.0000	64.000
	0.6931	10.000	6.9310	0.4804	100.000
	1.0986	9.000	9.8874	1.2069	81.000
	1.3863	8.000	11.0904	1.9218	64.000
	1.6094	7.000	11.2658	2.5902	49.000
	1.7918	6.000	10.7508	3.2105	36.000
	1.9459	6.000	11.6754	3.7865	36.000
	2.0794	3.000	6.2382	4.3239	9.000
	2.1972	2.000	4.3944	4.8277	4.000
SUM	12.8017	59.0000	72.2334	22.3479	443.000
MEAN	1.42241	6.55555			

Computation of Estimators:

$$\hat{b} = \frac{SXY - SX*SY/n}{SX2 - SX*SX/n}$$

$$= \frac{72.2334 - 12.8017 * 59/9}{22.3479 - 12.8017 * 12.8017/9} = \frac{-11.688855}{4.13862}$$

$$= -2.824336$$

$$\hat{a} = \bar{Y} - \hat{b} * \bar{X}$$

$$= 6.55555 - (-2.824336 * 1.42241)$$

$$= 10.572913$$

Total S.S. = SY2 - SY * SY/n

= 443 - 59 * 59/9

= 56.2222

Regression S.S.= $\hat{b}$ * SXY - SX * SY/n

= - 2.824336 * (72.2334 - 12.8017 * 59/9)

= 33.0132

Residual S.S. = Total S.S. - Regression S.S.

= 56.2222 - 33.0132

= 23.209

Tabulation of Linear Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	33.0132	1	33.0132	9.95
Residual	23.2090	7	3.315571	
Total	56.2222	8		

Computation of Reliability and Student's T test:

$$R\,2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{33.0132}{56.2222} * 100$$

$$= 58.72\%$$

$$Sb = \sqrt{\frac{\text{Residual M.S.}}{SX2 - SX * SX/n}}$$

$$Sb = \sqrt{\frac{3.315571}{22.3483 - 12.8017 * 12.8017/9}}$$

$$= 0.895058$$

$$t = \frac{\hat{b}}{Sb}$$

$$= -\frac{2.824336}{0.895058}$$

$$= -3.15$$

Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$:

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i)$$

$$\hat{Y}_1 = 10.572913 - 2.824336 * (0) = 10.573$$

$$\hat{Y}_2 = 10.572913 - 2.824336 * (0.6931) = 8.615$$

$$S\hat{Y}_i = \sqrt{\text{Residual M.S.} * \left\{\frac{1}{n} + \frac{(X_i - X)^2}{SX2 - SX * SX/n}\right\}}$$

$$S\hat{Y}_1 = \sqrt{3.315571 * \left\{\frac{1}{9} + \frac{(0.0 - 1.42241)^2}{22.3483 - 12.8017 * 12.8017/9}\right\}}$$

$$= \sqrt{\frac{3.315571}{9} + \frac{2.02325 * 3.315571}{4.13862}}$$

$$= 1.4104$$

$$S\hat{Y}_2 = \sqrt{3.31557 * \left\{\frac{1}{9} + \frac{(0.6931 - 1.42241)^2}{4.13862}\right\}}$$

$$= \sqrt{\frac{3.31557}{9} + \frac{0.531893 * 3.315571}{4.13862}}$$

$$= 0.8914$$

$$E_i = t * S\hat{Y}_i$$

$$E_1 = 2.365 * 1.4104 = 3.3356$$

$$E_2 = 2.365 * 0.8914 = 2.1082$$

Computation of Confidence Limits :

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 10.5729 - 3.3356 = 7.237$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 10.5729 + 3.3356 = 13.909$$

Tabulation of Observed, Estimated and Confidence Limits

var-X	var-Y	Y-Hat	Error	Confidence Limits Lower	Confidence Limits Upper
0.000	8.000	10.573	3.336	7.237	13.909
0.693	10.000	8.615	2.108	6.507	10.723
1.099	9.000	7.470	1.591	5.879	9.061
1.386	8.000	6.658	1.438	5.220	8.095
1.609	7.000	6.027	1.489	4.538	7.516
1.792	6.000	5.512	1.635	3.878	7.147
1.946	6.000	5.077	1.813	3.264	6.890
2.079	3.000	4.700	1.999	2.701	6.699
2.197	2.000	4.367	2.180	2.188	6.547

Scatter Diagram and Plot of Semilogarithm Regression :

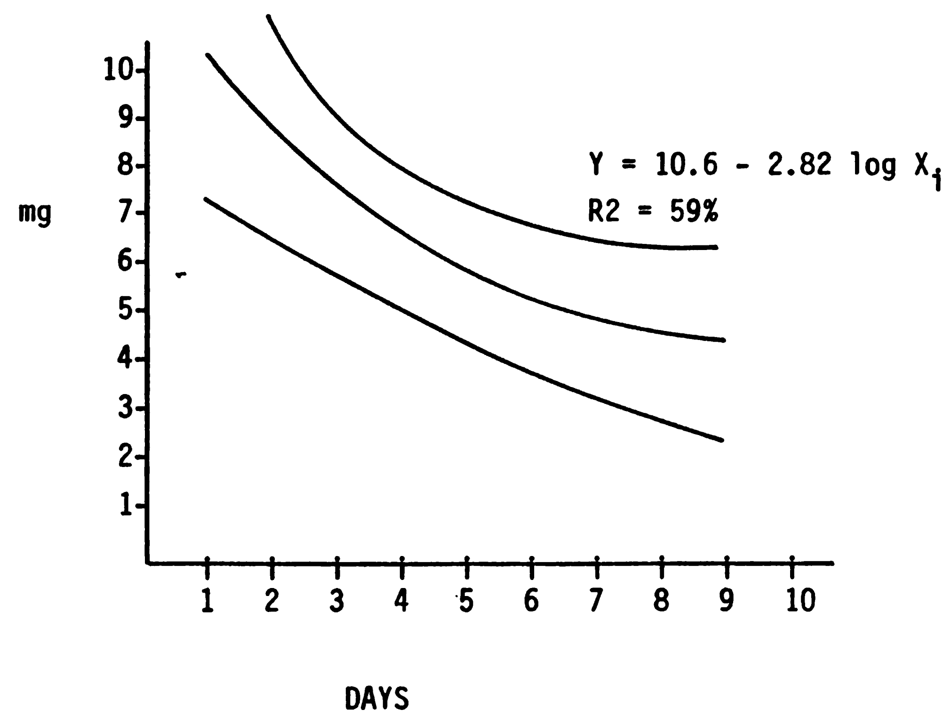

Figure 9. Semilogarithm Regression of concentration on days.

Logarithm Regression

$$Y_i = aX^b_i$$

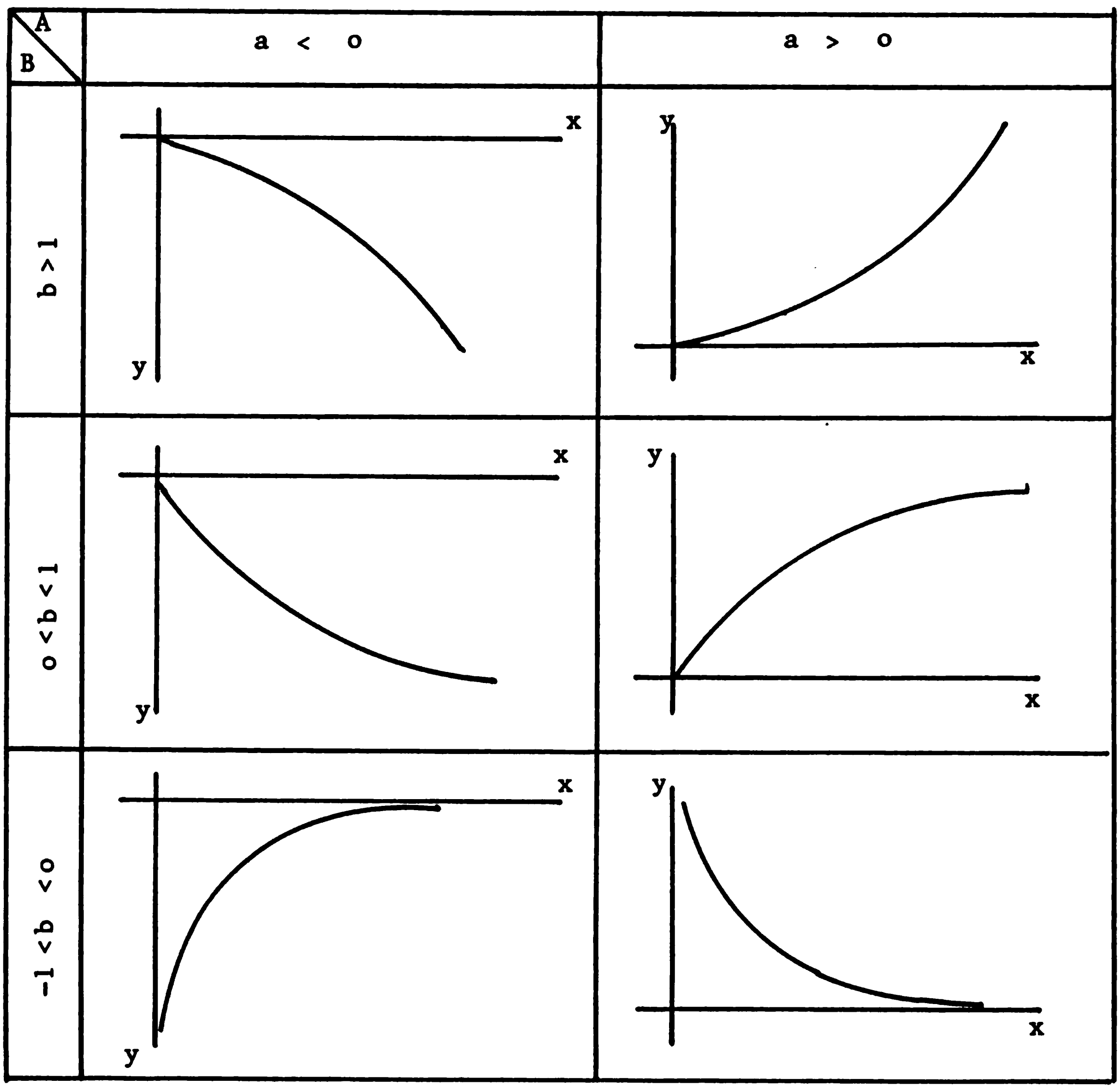

Figure 10. Spectrum of Logarithm Regression according to values a and b

EXAMPLE 3.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Establish a first order nonlinear regression for the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the logarithm model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limit in tabular form.

Tabulation of Sum Squares and Cross Products

	X_i	Y_i	$X_i * Y_i$	$X_i ** 2$	$Y_i ** 2$
	1.0000	8.000	8.0000	1.0000	64.000
	0.5000	10.000	5.0000	0.2500	100.000
	0.3333	9.000	2.9997	0.1111	81.000
	0.2500	8.000	2.0000	0.0625	64.000
	0.2000	7.000	1.4000	0.0400	49.000
	0.1667	6.000	1.0002	0.0278	36.000
	0.1429	6.000	0.8574	0.0204	36.000
	0.1250	3.000	0.3750	0.0156	9.000
	0.1111	2.000	0.2222	0.0123	4.000
SUM	2.8290	59.000	21.8545	1.5397	443.000
MEAN	0.31433	6.55555			

Computation of Estimators:

$$\hat{b} = \frac{SXY - SX*SY/n}{SX2 - SX*SX/n}$$

$$= \frac{21.8545 - 2.829 * 59/9}{1.5397 - 2.829 * 2.829/9} = \frac{3.308834}{0.650451}$$

$$= 5.086984$$

$$\hat{a} = \bar{Y} - \hat{b} * \bar{X}$$

$$= 6.55555 - (5.086984 * 0.31433)$$

$$= 4.956558$$

Total S.S. = SY2 - SY * SY/n

= 443 - 59 * 59/9

= 56.2222

Regression S.S.= $\hat{b}$ * SXY - SX * SY/n

= 5.086984 * (21.8545 - 2.829 * 59/9)

= 16.8320

Residual S.S. = Total S.S. - Regression S.S.

= 56.2222 - 16.8320

= 39.3902

Tabulation of Reciprocal Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression...........	16.8320	1	16.8320	2.99
Residual	39.3902	7	5.627171	
Total	56.2222	8		

Computation of Reliability and Student's T test:

$$R\ 2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{16.832}{56.2222} * 100$$

$$= 29.94\%$$

$$Sb = \sqrt{\frac{\text{Residual M.S.}}{SX2 - SX * SX/n}}$$

$$= \sqrt{\frac{5.627171}{1.54 - 2.829 * 2.829/9}}$$

$$= 2.941289$$

$$t = \frac{\hat{b}}{Sb}$$

$$= \frac{5.086984}{2.941289}$$

$$= 1.73$$

Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$:

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i)$$

$$\hat{Y}_1 = 4.956558 + 5.086984 * (1) = 10.043$$

$$\hat{Y}_2 = 4.956558 + 5.086984 * (0.5) = 7.500$$

$$S\hat{Y}_i = \sqrt{\text{Residual M.S.} * \left\{\frac{1}{n} + \frac{(X_i - \bar{X})^2}{SX2 - SX * SX/n}\right\}}$$

$$S\hat{Y}_1 = \sqrt{5.627171 * \left\{\frac{1}{9} + \frac{(1 - 0.31433)^2}{0.650451}\right\}}$$

$$= \sqrt{\frac{5.627171}{9} + \frac{5.627 * 0.470143}{0.650451}}$$

$$= 2.166197$$

$$S\hat{Y}_2 = \sqrt{5.627171 * \left\{\frac{1}{9} + \frac{(0.5 - 0.31433)^2}{0.650451}\right\}}$$

$$= \sqrt{\frac{5.627171}{9} + \frac{5.627171 * 0.034473}{0.650451}}$$

$$= 0.960975$$

$$E_i = t * S\hat{Y}_i$$

$$E_1 = 2.365 * 2.166197 = 5.123$$

$$E_2 = 2.365 * 0.960975 = 2.273$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 10.043 - 5.12 = 4.923$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 10.043 + 5.12 = 15.163$$

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
1.000	8.000	10.043	5.123	4.923	15.163
0.500	10.000	7.500	2.273	5.227	9.773
0.333	9.000	6.652	1.875	4.777	8.527
0.250	8.000	6.228	1.923	4.305	8.151
0.200	7.000	5.974	2.032	3.942	8.006
0.167	6.000	5.804	2.134	3.671	7.938
0.143	6.000	5.683	2.218	3.465	7.901
0.125	3.000	5.593	2.287	3.305	7.880
0.111	2.000	5.522	2.344	3.178	7.866

Scatter Diagram and Plot of Reciprocal Regression;

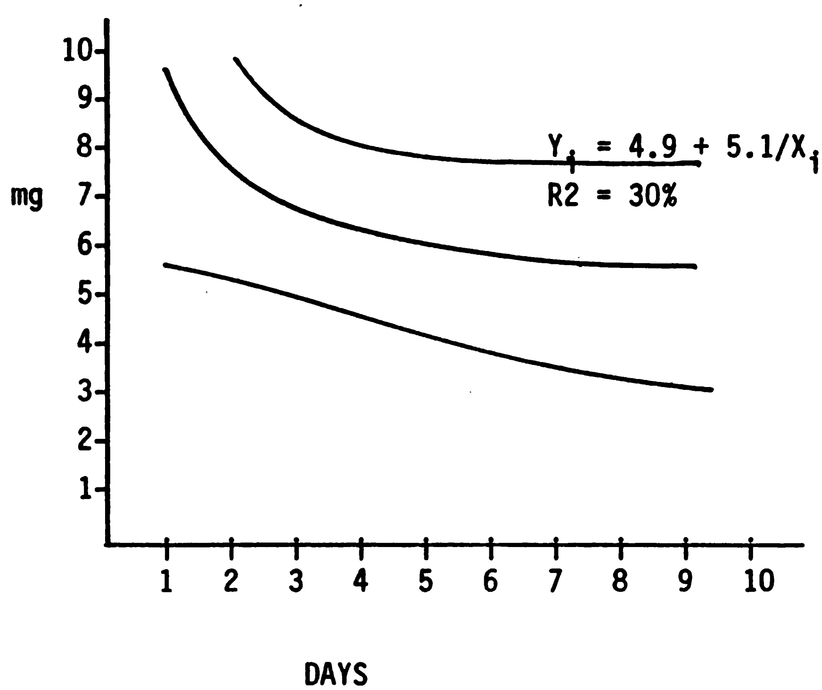

Figure 15. Reciprocal Regression of concentration of drugs on days.

Five examples of Second Order Linear Models

Quadratic Regression

$$Y_i = a + bX_i + cX^2_i$$

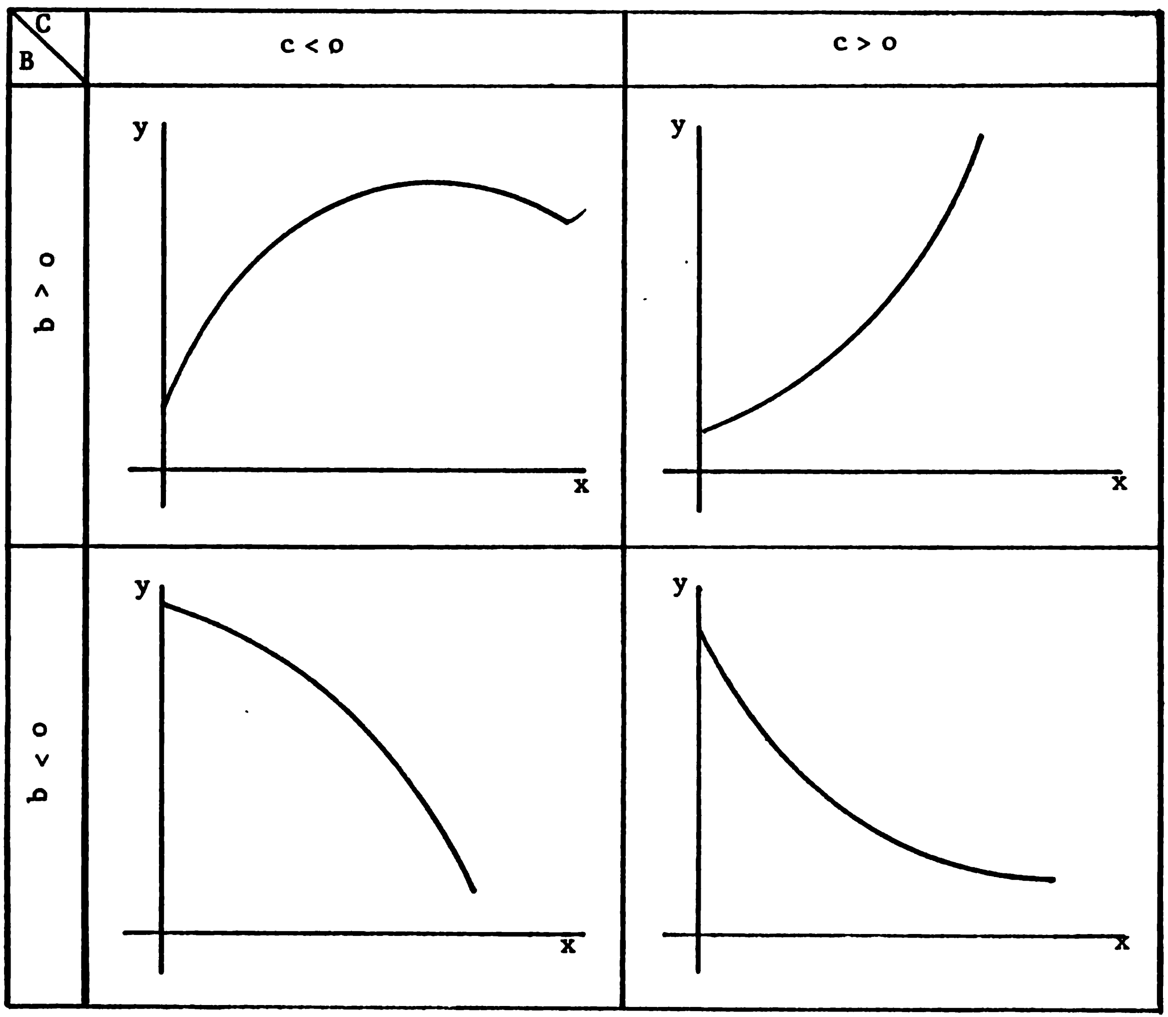

Figure 16. Spectrum of Quadratic Regression according to values of $\hat{a}$, $\hat{b}$, and $\hat{c}$.

EXAMPLE 6.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Fit a second order regression model to the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the quadratic regression model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limits in tabular form.

Tabulation of Sum Squares and Cross Products

	var-X_i	var-Y_i	var-Z_i	X_i**2	X_i**3	X_i**4	X_i*Y_i	Z_i*Y_i	Y_i**2
	1.0	8.0	1.0	1	1	1	8	8	64
	2.0	10.0	4.0	4	8	16	20	40	100
	3.0	9.0	9.0	9	27	81	27	81	81
	4.0	8.0	16.0	16	64	256	32	128	64
	5.0	7.0	25.0	25	125	625	35	175	49
	6.0	6.0	36.0	36	216	1296	36	216	36
	7.0	6.0	49.0	49	343	2401	42	294	36
	8.0	3.0	64.0	64	512	4096	24	192	9
	9.0	2.0	81.0	81	729	6561	18	162	4
SUM:	45.0	59.0	285.0	285	2025	15333	242	1296	443
MEAN:	5.0	6.5555	31.6666						

Computation of Estimators:

$$\hat{b} = \frac{(SXY - SX*SY/n)(SX4 - SZ*SZ/n) - (SZY - SZ*SY/n)(SX3 - SZ*SX/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)(SX3 - SZ*SX/n)}$$

$$= \frac{(242 - 45*59/9)(15333 - 285*285/9) - (1296 - 285*59/9)(2025 - 285*45/9)}{(285 - 45*45/9)(15333 - 285*285/9) - (2025 - 285*45/9)(2025 - 285*45/9)}$$

$$= \frac{(-53)(6308) - (-572.3333)(600)}{(60)(6308) - (600)(600)}$$

$$= 0.491125$$

$$\hat{c} = \frac{(SX2 - SX*SX/n)(SZY - SZ*SY/n) - (SZ3 - SZ*SY/n)(SXY - SX*SY/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)(SX3 - SZ*SX/n)}$$

$$=\frac{(285 - 45*45/9)(1296 - 285*59/9) - (2025 - 285*45/9)(242 - 45*59/9)}{(285 - 45*45/9)(15333 - 285*285/9) - (2025 - 285*45/9)(2025 - 285*45/9)}$$

$$= \frac{(60)(-572.3333) - (600)(-53)}{(60)(6308) - (600)(600)}$$

$$= -0.137446$$

$$\hat{a} = \bar{Y} - \hat{b}\,\bar{X} - \hat{C}Z^2$$

$$= 6.55555 - (0.491) * (5) - (-0.137446) * (31.6666)$$

$$= 8.452381$$

Total S.S. = SY2 - SY * SY/n

= 443 - 59 * 59/9

= 56.2222

Regression S.S.

$= \hat{b}*(SXY - SX*SY/n) + \hat{c}* (SZY - SZ*SY/n)$

= .491125 * (242 - 45*59/9) + (- 0.137446) * (1296 - 285*59/9)

= 52.6353

Residual S.S. = Total S.S. - Regression S.S.

= 56.2222 - 52.6353

= 3.5869

Tabulation of Quadratic Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	52.6353	2	26.318	44.02
Residual	3.5869	6	0.59782	
Total	56.2222	8		

Computation of Reliability and Student's T test:

$$R2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{52.6353}{56.2222} * 100$$

$$= 93.62\%$$

$$sb = \sqrt{rMS.* \frac{SX4 - SZ*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)^2}}$$

$$sb = \sqrt{0.59782 * \frac{6308}{18480}}$$

$$sb = 0.451731$$

$$sc = \sqrt{rMS.* \frac{SX2 - SX*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)^2}}$$

$$sc = \sqrt{0.59782 * \frac{60}{18480}}$$

$$sc = 0.044056$$

$$sbc = \sqrt{rMS. * \frac{SX3 - SZ*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)^2}}$$

$$sbc = \sqrt{0.59782 * \frac{600}{18480}}$$

$$sbc = 0.139319$$

Where: sb = Standard error of $\hat{b}$

sc = Standard error of $\hat{c}$

sbc = Square root of covariance between $\hat{b}$ and $\hat{c}$

$$t = \frac{\hat{b}}{sb} = \frac{0.491125}{0.451731} = 1.09$$

$$t = \frac{\hat{c}}{sc} = \frac{0.137446}{0.044056} = - 3.12$$

<u>Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$:</u>

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i) + \hat{c} * (Z_i)$$

$$\hat{Y}_1 = 8.452381 + 0.491125 * (1) + (-0.137446) * (1) = 8.806$$

$$\hat{Y}_2 = 8.452381 + 0.491125 * (2) + (-0.137446) * (4) = 8.885$$

$$SY_i = \sqrt{\frac{rMs}{n} + (sb)^2(X_i - \bar{X})^2 + (sc)^2 (Z_i - \bar{Z})^2 - 2(sbc)^2 (X_i - \bar{X})(Z_i - \bar{Z})}$$

$$SY_1 = \sqrt{\frac{0.59782}{9} + (0.451731)^2 (1-5)^2 + (0.044056)^2 (1-31.66666)^2 \ldots}$$

$$\ldots - 2(0.139319)^2 (1-5)(1-31.66666)$$

$$= \sqrt{0.066436 + 3.264973 + 1.825119 - 4.761845} = 0.628228$$

$$SY_2 = \sqrt{\frac{0.59782}{9} + (0.451731)^2 (2-5)^2 + (0.044056)^2 (4-31.66666)^2 \ldots}$$

$$\ldots - 2(0.139319)^2 (2-5)(4 - 31.66666)$$

$$= \sqrt{0.066436 + 1.836547 + 1.485497 - 3.222009} = 0.408008$$

$$E_i = t * SY_i$$

$$E_1 = 2.447 * 0.628228 = 1.537$$

$$E_2 = 2.447 * 0.408008 = 0.998$$

<u>Computation of Confidence Limits</u>:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 8.806 - 1.537 = 7.269$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 8.806 + 1.537 = 10.343$$

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
1.000	8.000	8.806	1.538	7.268	10.344
2.000	10.000	8.885	0.999	7.886	9.884
3.000	9.000	8.689	0.848	7.841	9.537
4.000	8.000	8.218	0.911	7.306	9.129
5.000	7.000	7.472	0.956	6.516	8.428
6.000	6.000	6.451	0.911	5.540	7.362
7.000	6.000	5.155	0.848	4.307	6.003
8.000	3.000	3.585	0.999	2.586	4.584
9.000	2.000	1.739	1.538	0.202	3.277

Scatter Diagram and Plot of Quadratic Regression:

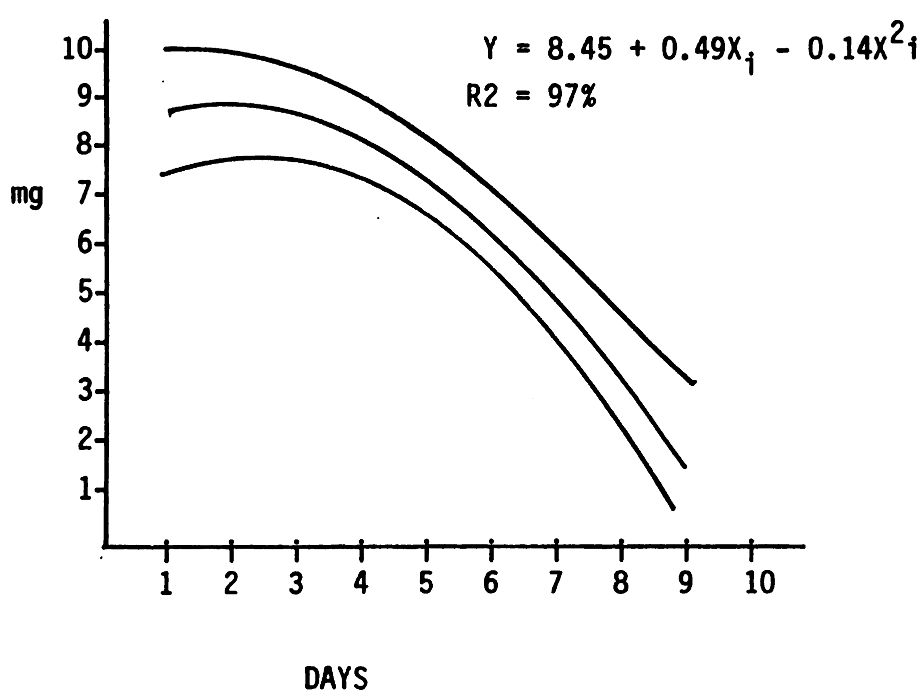

Figure 17. Quadratic Regression of Concentration of drugs on days.

Square Root Regression

$$Y_i = a + bX_i + cX_i^{0.5}$$

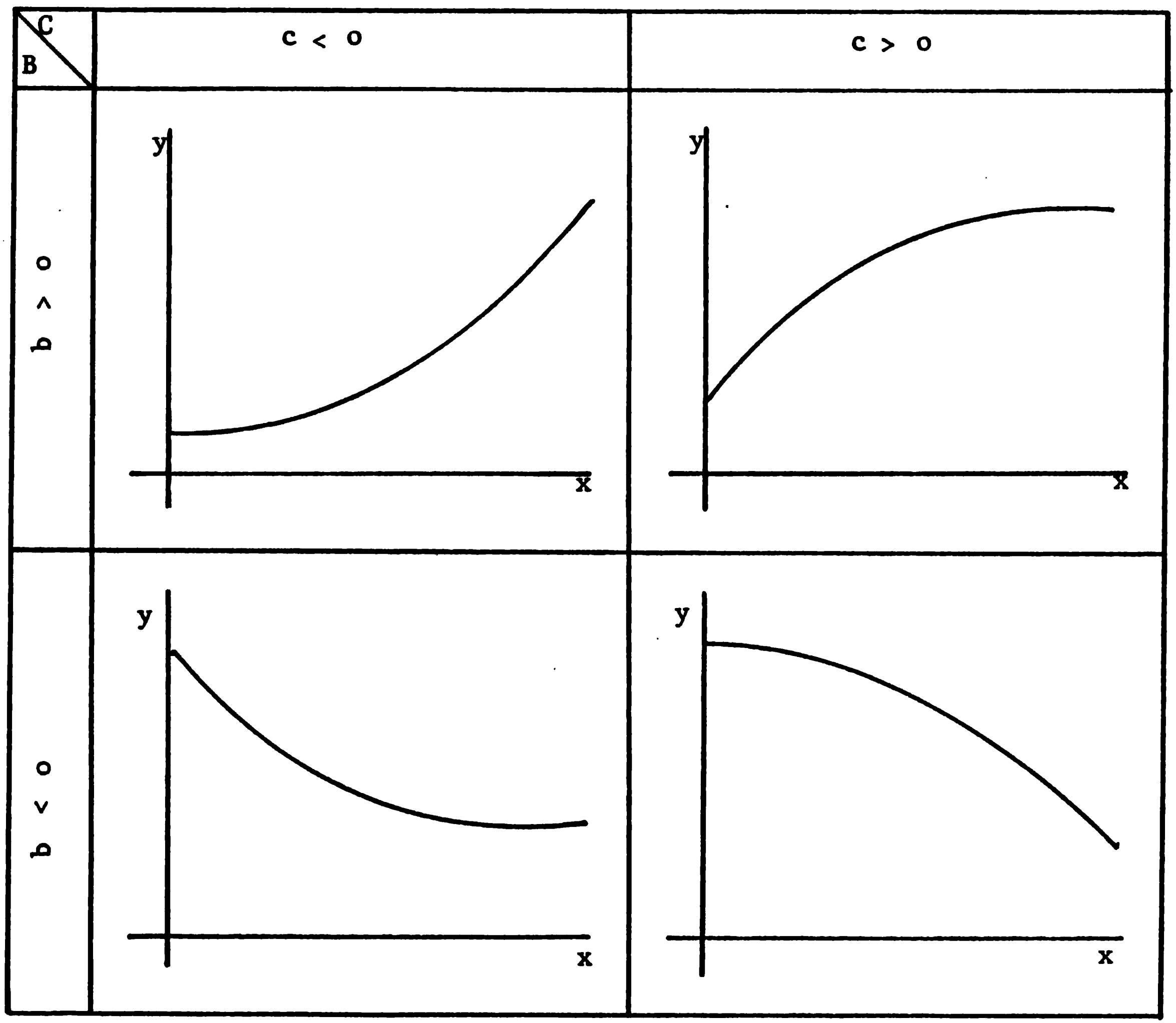

Figure 18. Spectrum of Square Root Regression according to values of a, b and c

EXAMPLE 7.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Fit a second order regression model to the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the square root regression model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limits in tabular form.

Tabulation of Sum Squares and Cross Products

	var-X_i	var-Y_i	var-Z_i	X_i**2	X_i**3	X_i**4	X_i*Y_i	Z_i*Y_i	Y_i**2
	1	8	1.0000	1	1.0000	1	8	8.0000	64
	2	10	1.4142	4	2.8284	2	20	14.1421	100
	3	9	1.7321	9	5.1962	3	27	15.5885	81
	4	8	2.0000	16	8.0000	4	32	16.0000	64
	5	7	2.2361	25	11.1803	5	35	15.6525	49
	6	6	2.4495	36	14.6969	6	36	14.6969	36
	7	6	2.6458	49	18.5203	7	42	15.8745	36
	8	3	2.8284	64	22.6274	8	24	8.4853	9
	9	2	3.0000	81	27.0000	9	18	6.0000	4
SUM:	45	59	19.3060	285	111.0495	45	242	114.4398	443
MEAN:	5	6.5555	2.14511						

Computation of Estimators :

$$\hat{b} = \frac{(SXY - SX*SY/n)(SX4 - SZ*SZ/n) - (SZY - SZ*SY/n)(SX3 - SZ*SX/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)(SX3 - SZ*SX/n)}$$

$$= \frac{(242 - 45*59/9)(45 - 19.3060 * 193060/9)-}{(285 - 45*45/9)(45 - 19.3060 * 193060/9)-} \ \ldots$$

$$\ldots \frac{(\ 114.4398 - 19.3060 * 59/9)(111.0495 - 19.3060 * 45/9}{(111.0495 - 19.3060 * 59/9)(111.0495 - 19.3060 * 45/9}$$

$$= \frac{(-53)\ (3.586486) - (12.12175)\ (14.5195}{(60)\ (3.586486) - (14.5195)\ \ (14.5195}$$

$$= -\ 3.220011$$

$$\hat{c} = \frac{(SX2 - SX*SX/n)(SZY - SZ*SY/n) - (SX3 - SX*SZ/n)(SXY - SX*SY/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SZ*SX/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(285 - 45*45/9)(114.4398 - 19.306*59/9) -}{(285 - 45*45/9)(45 - 19.306/9) -} \ldots$$

$$\ldots \frac{(111.0495 - 45*19.306/9)(111.0495 - 45*19.306/9)}{(111.0495 - 45*19.306/9)(111.0495 - 45*19.306/9)}$$

$$= \frac{(60)\ (-12/12175) - (14.5195)\ (-\ 53)}{(60)(3.586486) - (14.5195)\ (14.5195)}$$

$$= 9.656025$$

$$\hat{a} = \bar{Y} - \hat{b}\,\bar{X} - \hat{C}\bar{Z}$$

$$= 6.55555 - (-3.220011)(5) - (9.656025)\ (2.14511)$$

$$= 1.942365$$

Total S.S. = SY2 - SY * SY/n

= 443 - 59 * 59/9

= 56.2222

Regression S.S

$$= \hat{b}*(SXY - SX*SY/n) + \hat{c}*\ (SZY - SZ*SY/n)$$

$$= -3.220011\ (242 - 45*59/9) + 9.656025\ (114.4398 - 19.306*59/9)$$

$$= 53.6127$$

Residual S.S. = Total S.S. - Regression S.S.

= 56.2222 - 53.6127

= 2.6095

Tabulation of Root Square Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	53.6127	2	26.8064	61.64
Residual	2.6095	6	0.43492	
Total	56.2222	8		

Computation of Reliability and Student's T test:

$$R2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{53.6127}{56.2222} * 100$$

$$= 95.36\%$$

$$sb = \sqrt{rMS. * \frac{SX4 - SZ*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$sb = \sqrt{0.43492 * \frac{3.586486}{4.37328}}$$

$$= 0.597222$$

$$sc = \sqrt{rMS.* \frac{SX2 - SX*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{\frac{0.43492 * 60}{4.37328}}$$

$$= 2.442737$$

$$sbc = \sqrt{rMS.* \frac{SX3 - SX*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.43492 * \frac{14.5195}{4.37328}}$$

$$= 1.201647$$

$$t = \frac{\hat{b}}{sb} = \frac{-3.220011}{0.597222} = - 5.39$$

$$t = \frac{\hat{c}}{sc} = \frac{9.651025}{2.442737} = 3.95$$

<u>Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$:</u>

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i) + \hat{c}* (Z_i)$$

$$\hat{Y}_1 = 1.942365 + (-3.220011)(1) + (9.656025)(1) = 8.378$$

$$\hat{Y}_2 = 1.942365 + (-3.220011)(2) + (9.656025)(1.4142) = 9.158$$

$$SY_i = \sqrt{\frac{rMs}{n} + (sb)^2(X_i - \bar{X})^2 + (sc)^2 (Z_i - \bar{Z})^2 - 2(sbc)^2 (X_i - \bar{X})(Z_i - \bar{Z})}$$

$$SY_1 = \sqrt{\frac{0.43492}{9} + (0.597222)^2 (1-5)^2 + (2.442737)^2 (1 - 2.14511)^2 \ldots}$$

$$\ldots - 2(1/201647)^2 (1-5)(1-2.145111)$$

$$= \sqrt{0.0048324 + 5.706805 + 7.824342 - 13.227903} = 0.592932$$

$$SY_2 = \sqrt{\frac{0.43492}{9} + (0.597222)^2(2-5)^2 + (2.442737)^2(1.4142 - 2.14511)^2 \ldots}$$

$$\ldots - 2(1.201647)^2 (2-5)(1.4142 - 2.14511)$$

$$= \sqrt{0.048324 + 3.210067 + 3.187727 - 6.332409} = 0.337208$$

$$E_i = t * SY_i$$

$$E_1 = 2.447 * 0.592932 = 1.450905$$

$$E_2 = 2.447 * 0.337208 = 0.825148$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 8.378 - 1.450905 = 6.927$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 8.378 + 1.450905 = 9.829$$

Tabulation of Observed, Estimated and Confidence Limits

var-X	var-Y	Y-Hat	Error	Confidence Limits Lower	Upper
1.000	8.000	8.378	1.450	6.928	9.828
2.000	10.000	9.158	0.825	8.333	9.983
3.000	9.000	9.007	0.801	8.206	9.809
4.000	8.000	8.375	0.817	7.557	9.192
5.000	7.000	7.434	0.765	6.669	8.199
6.000	6.000	6.275	0.687	5.588	6.961
7.000	6.000	4.950	0.687	4.262	5.637
8.000	3.000	3.494	0.863	2.631	4.356
9.000	2.000	1.930	1.200	0.730	3.130

Scatter Diagram and Plot of Square Root Regression :

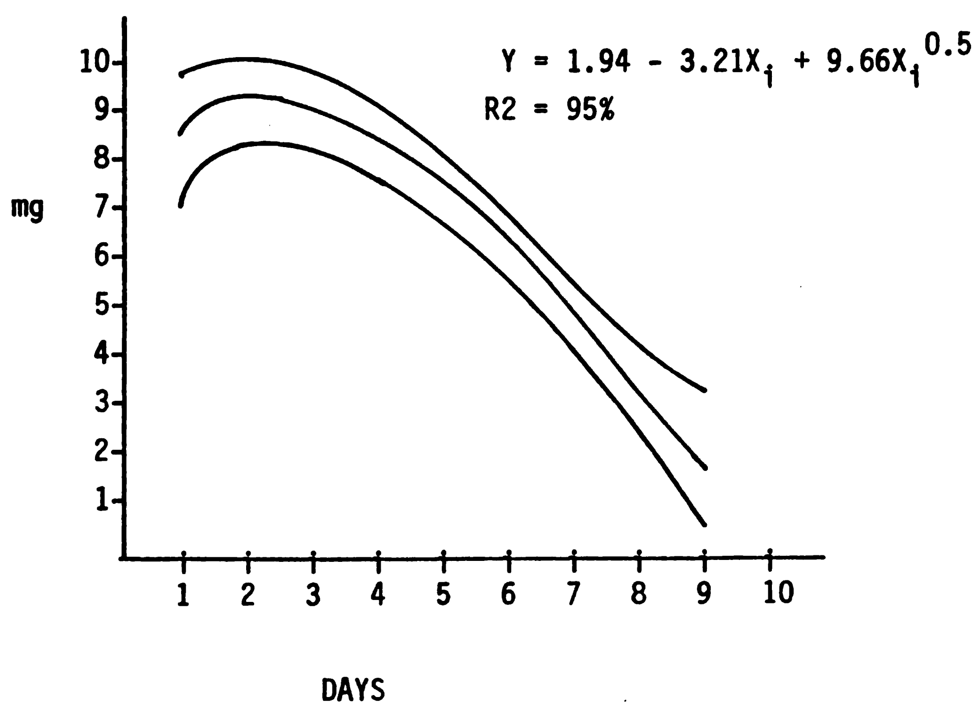

Figure 19. Square Root Regression of concentration of drugs on days.

Gamma Regression Model

$$Y_i = a\,e^{bX_i}\,X_i^{c}$$

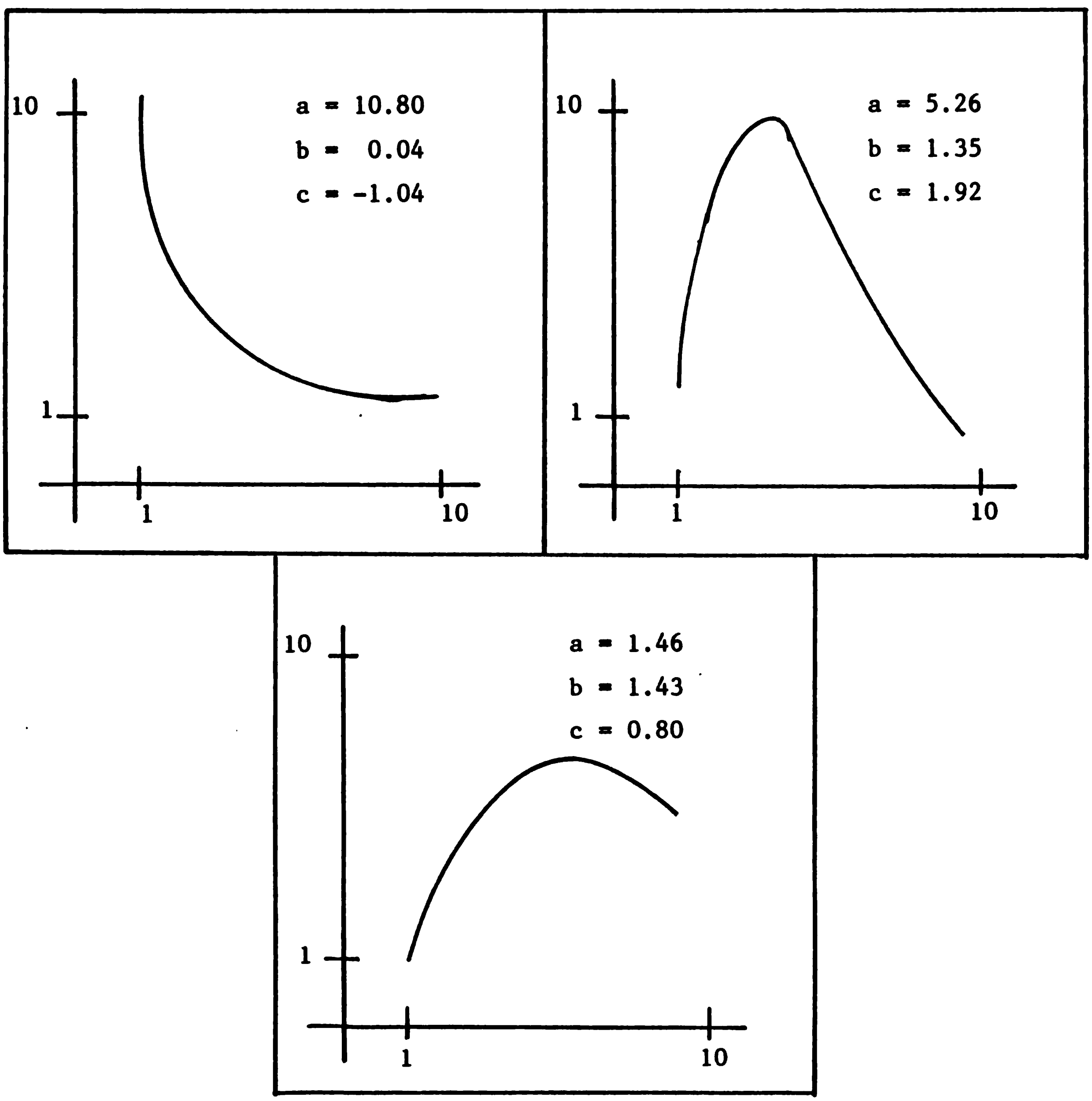

Figure 20. Spectrum of Gamma Regression according to values of $\hat{a}$, $\hat{b}$ and $\hat{c}$.

EXAMPLE 8.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Fit a second order regression model to the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the gamma regression model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limits in tabular form.

Tabulation of Sum Squares and Cross Products

	var-X_i	var-Y_i	var-Z_i	X_i**2	X_i**3	X_i**4	X_i*Y_i	Z_i*Y_i
	1	2.07944	0.00000	1	0.0000	0.00000	2.0794	0.00000
	2	2.03259	0.69315	4	1.3863	0.48045	4.6052	1.59603
	3	2.19722	1.09861	9	3.2958	1.20695	6.5917	2.41390
	4	2.-7944	1.38629	16	5.5452	1.92181	8.3178	2.88272
	5	1.94591	1.60944	25	8.0472	2.59029	9.7296	3.13182
	6	1.79176	1.79176	36	10.7506	3.21040	10.7506	3.21040
	7	1.79176	1.94591	49	13.6214	3.78657	12.5423	3.48660
	8	1.09861	2.07944	64	16.6355	4.32408	8.7889	2.28450
	9	0.69315	2.19722	81	19.7750	4.82780	6.2383	1.52300
SUM:	45	15.97988	12.80182	285	79.0570	22.34835	69.6438	20.52897
MEAN:	5	1.77553	1.422424					

Computation of Estimators:

$$\hat{b} = \frac{(SXY - SX*SY/n)(SX4 - SZ*SZ/n) - (SZY - SZ*SY/n)(SX3 - SX*SZ/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(69.6438 - 45*15.97988/9)(22.34835 - 12.80182 * 12.80182/9) - \ldots}{(285 - 45*45/9)(22.34835 - 12.80182 * 12.80182/9) -}$$

$$= \frac{\ldots(20.52897 - 12.80182 * 15.97988/9)(79.057 - 45 * 12.80182/9}{(79.057 - 45 * 12.80182/9)(79.057 - 45 * 12.80182/9}$$

$$= \frac{(-10.2556)(4.138729) - (-2.201201)(15.0479)}{(60)\ (4.138729) - (15.0479)\ (15.0479)} = - \frac{9.321697}{21.88445}$$

$$= - 0.425951$$

$$\hat{c} = \frac{(SX2 - SX*SX/n)(SZY - SZ*SY/n) - (SX3 - SX*SZ/n)(SXY - SX*SY/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(285 - 45*45/9)(20.52897 - 12.80182 * 15.97988/9) - \quad ...}{(285 - 45*45/9)(22.34835 - 12.80182 * 12.80182/9) -}$$

$$... \frac{(79.057 - 45 * 12.80182/9)(69.6438 - 45 * 15.97988/9)}{(79.057 - 45 * 12.80182/9)(79.057 - 45 * 15.97988/9)}$$

$$= \frac{(60)(-2.201201) - (15.0479)(-10.2556)}{(60)(4.138729) - (15.0479)(15.0479)} = \frac{22.25318}{21.88445}$$

$$= 1.016849$$

$$\hat{a} = \bar{Y} - \hat{b}\bar{X} - \hat{c}\bar{Z}$$

$$= 1.77553 - (-0.425951)(5) - (1.016849)(1.422424)$$

$$= 2.458895$$

Total S.S. $= SY2 - SY * SY/n$

$= 30.6726 - 15.97988 * 15.97988/9$

$= 2.29965$

Regression S.S

$= \hat{b}*(SXY - SX*SY/n) + \hat{c}*(SZY - SZ*SY/n)$

$= -0.425951 (-10.2556) + 1.016849 (-2.201201)$

$= 2.1301$

Residual S.S. = Total S.S. - Regression S.S.

= 2.2996 - 2.1301

= 0.1695

Tabulation of Gamma Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	2.1301	2	1.0650	37.70
Residual	0.1695	6	0.02825	
Total	2.2996	8		

Computation of Reliability and Student's T test:

$$R2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{2.1301}{2.2996} * 100$$

$$= 93.63\%$$

$$sb = \sqrt{rMS.* \frac{SX4 - SZ*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$sb = \sqrt{0.02825 * \frac{4.138729}{21.88445}}$$

$$= 0.073092$$

$$sc = \sqrt{rMS.* \frac{SX2 - SX*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.02825 * \frac{60}{21.88445}}$$

$$= 0.278302$$

$$sbc = \sqrt{rMS.* \frac{SX3 - SX*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.02825 * \frac{15.0479}{21.88445}}$$

$$= 0.139373$$

$$t = \frac{\hat{b}}{sb} = \frac{-0.425951}{0.073092} = 5.83$$

$$t = \frac{\hat{c}}{sc} = \frac{1.016849}{0.278302} = 3.65$$

Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i) + \hat{c} * (Z_i)$$

$$\hat{Y}_1 = 2.458895 + (-0.425951)(1) + 1.016849\ (0) = 2.033$$

$$\hat{Y}_2 = 2.458895 + (-0.425951)\ (2) + 1.016849\ (0.69315) = 2.312$$

$$SY_i = \sqrt{\frac{rMs}{n} + (sb)^2(X_i - \bar{X})^2 + (sc)^2\ (Z_i - \bar{Z})^2 - 2(sbc)^2(X_i - \bar{X})(Z_i - \bar{Z})}$$

$$SY_1 = \sqrt{\frac{0.02825}{9} + (0.073092)^2\ (1-5)^2 + (0.278302)^2\ (0 - 1.422424)^2 \ldots}$$

$$\ldots - 2(0.139373)^2\ (1-5)(0 - 1.422424)$$

$$= \sqrt{0.003139 + 0.085478 + 0.156708 - 0.221042 = 0.155829}$$

$$SY_2 = \sqrt{\frac{0.02825}{9} + (0.073092)^2(2-5)^2 + (0.278302)^2(0.6931 - 1.422424)^2 \ldots}$$

$$\ldots - 2(0.139373)^2\ (2-5)(0.6931 - 1.422424)$$

$$= \sqrt{0.003139 + 0.048082 + 0.041198 - 0.085 = 0.086134}$$

$$E_i = t*SY_i$$

$$E_1 = 2.447 * 0.155829 = 0.381$$

$$E_2 = 2.447 * 0.086134 = 0.211$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 2.033 - 0.381 = 1.652$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 2.033 + 0.381 = 2.414$$

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
1.000	2.079	2.033	0.381	1.652	2.414
2.000	2.303	2.312	0.211	2.101	2.523
3.000	2.197	2.298	0.212	2.087	2.510
4.000	2.079	2.165	0.207	1.957	2.372
5.000	1.946	1.966	0.187	1.779	2.153
6.000	1.792	1.725	0.168	1.557	1.893
7.000	1.792	1.456	0.174	1.282	1.630
8.000	1.099	1.166	0.220	0.946	1.386
9.000	0.693	0.860	0.297	0.563	1,156

Scatter Diagram and Plot of Gamma Regression

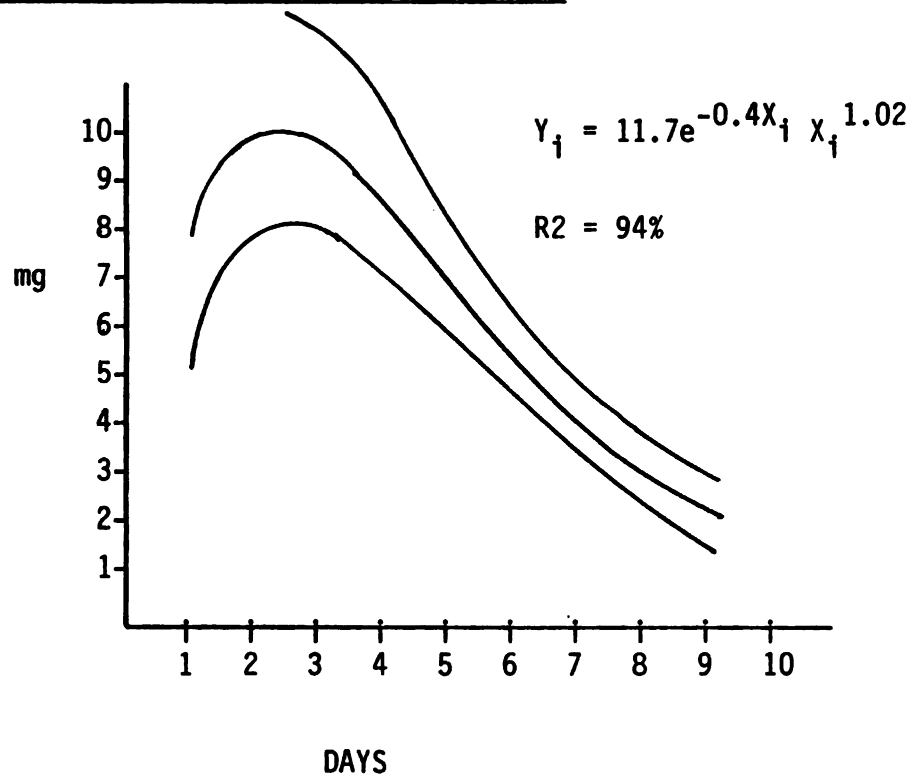

Figure 21. Gamma Regression of concentration of drugs on days.

<u>Beta Regression Model</u>

$$Y_i = a\, X_i^{\,b}\, (10 - X_i)^c$$

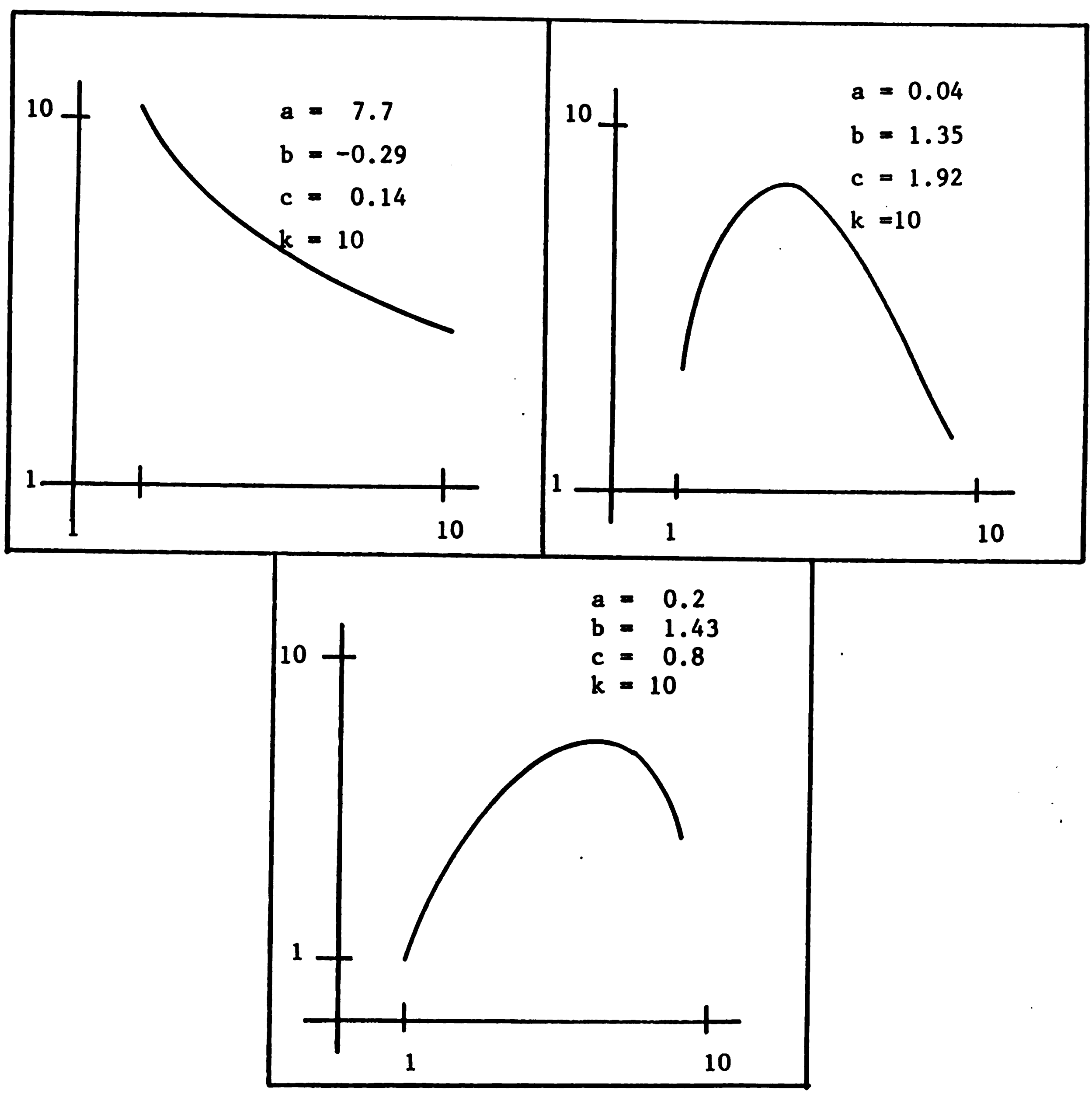

Figure 22. Spectrum of Beta Regression according to values of $\hat{a}$, $\hat{b}$ and $\hat{c}$.

EXAMPLE 9.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Fit a second order regression model to the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the beta regression model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limits in tabular form.

Tabulation of Sum Squares and Cross Products

	var-X_i	var-Y_i	var-Z_i	X_i**2	X_i**3	X_i**4	X_i*Y_i	Z_i*Y_i
	0.0000	2.0794	2.1972	0.0000	0.0000	4.8277	0.00000	4.5688
	0.6931	2.3026	2.0794	0.4804	1.4413	4.3239	1.59593	4.7880
	1.0986	2.1972	1.9459	1.2069	2.1378	3.7863	2.41384	4.2755
	1.3863	2.0794	1.7918	1.9218	2.4840	3.2106	2.88267	3.7259
	1.6094	1.9459	1.6094	2.5902	2.5902	2.5902	3.13173	3.1317
	1.7918	1.7918	1.3863	3.2105	2.4840	1.9218	3.21055	2.4840
	1.9459	1.7918	1.0986	3.7865	2.1378	1.2069	3.48664	1.9685
	2.0794	1.0986	0.6931	4.3239	1.4412	0.4804	2.28443	0.7614
	2.1972	0.6931	0.0000	4.8277	0.0000	0.0000	1.52288	0.0000
SUM:	12.8017	15.9798	12.8017	22.3479	14.7163	22.3480	20.52869	25.7038
MEAN:	1.42241	6.5555	1.42241					

Computation of Estimators:

$$\beta = \frac{(SXY - SX*SY/n)(SX4 - SZ*SZ/n) - (SZY - SZ*SY/n)(SX3 - SX*SZ/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(20.52869 - 12.8017*15.9798/9)(22.3480 - 12.8017*12.8017/9)-}{(22.3479 - 12.8017*12.8017/9)(22.3480 - 12.8017*12.8017/9)-} \ldots$$

$$\ldots \frac{(25.7038 - 12.8017*15.9798/9)(14.7163 - 12.8017*12.8017/9)}{(14.7163 - 12.8017*12.8017/9)(14.7163 - 12.8017*12.8017/9}$$

$$= \frac{(-2.201154)(4.13872) - (2.973956)(-3.49298)}{(4.13862)\ (4.13862) - (-3.49298)(-3.49298)}$$

$$= \frac{1.278008}{4.927266}$$

$$= 0.259375$$

$$\hat{c} = \frac{(SX2 - SX*SX/n)(SZY - SZ*SY/n) - (SX3 - SX*SZ/n)(SXY - SX*SY/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(22.3479 - 12.8017*12.8017/9)(25.7038 - 12.8017*15.9798/9) - \ldots}{(22.3479 - 12.8017*12.8017/9)(22.348 - 12.8017*12.8017/9) -}$$

$$\ldots \frac{(14.7163 - 12.8017*12.8017/9)(20.52869 - 12.8017*15.9798/9)}{(14.7163 - 12.8017*12.8017/9)(14.7163 - 12.8017*12.8017/9)}$$

$$= \frac{(4.13862)(2.973956) - (-3.49298)(-2.20454)}{(4.23862)(4.13862) - (-3.49298)(-3.49298)}$$

$$= \frac{4.619486}{4.927266}$$

$$= 0.937535$$

$$\hat{a} = \hat{Y} - \hat{b}\bar{X} - \hat{C}\bar{Z}$$

$$= 1.77553 - (0.259375)(1.42241) - (0.937535)(1.42241)$$

$$= 0.073033$$

Total S.S. $= SY2 - SY * SY/n$

$$= 30.6726 - 15.97988 * 15.97988/9$$

$$= 2.29965$$

Regression S.S

$$= \hat{b}*(SXY - SX*SY/n) + \hat{c}*(SZY - SZ*SY/n)$$

$$= (0.259375)(-2.201154) + (0.937535)(2.973956)$$

$$= 2.21726$$

Residual S.S. = Total S.S. - Regression S.S.

= 2.29965 - 2.21726

= 0.08239

Tabulation of Beta Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	2.21726	2	1.10863	80.73
Residual	0.08239	6	0.013732	
Total	2.29965	8		

Computation of Reliability and Student's T test:

$$R2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{2.21726}{2.29965} * 100$$

= 96.42%

$$sb = \sqrt{rMS.* \frac{SX4 - SZ*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.013732 * \frac{4.13872}{4.927266}}$$

$$= 0.107398$$

$$sc = \sqrt{rMS.* \frac{SX2 - SX*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ2*SZ2/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.013732 * - \frac{4.13862}{4.927266}}$$

$$= 0.107398$$

$$(sbc)^2 = \sqrt{rMS.* \frac{SX3 - SX*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.013732 * \frac{(-3.49298)}{4.927266}}$$

$$= 0.009735$$

$$t = \frac{\hat{b}}{sb} = \frac{0.259375}{0.107398} = 2.42$$

$$t = \frac{\hat{c}}{sc} = \frac{0.937535}{0.107398} = 8.73$$

Computation of $\hat{Y}_i$ Estimates and Standard Errors of $\hat{Y}_i$

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i) + \hat{c}* (Z_i)$$

$$\hat{Y}_1 = 0.073033 + (0.259375)(0) + (0.937535)(2.1972) = 2.133$$

$$\hat{Y}_2 = 0.073033 + (0.259375)(0.6931) + (0.937535)(2.0794) = 2.202$$

$$SY_i = \sqrt{\frac{rMs}{n} + (sb)^2(X_i - \bar{X})^2 + (sc)^2 (Z_i - \bar{Z})^2 - 2(sbc)^2(X_i - \bar{X})(Z_i - \bar{Z})}$$

$$SY_1 = \sqrt{\frac{0.013732}{9} + (0.107398)^2 (1.42241)^2 + (0.107398)^2 (0.77479)^2 \ldots}$$

$$\ldots - 2(-0.009735)(-1.42241)(0.77479)$$

$$= \sqrt{0.001526 + 0.023337 + 0.006926 - 0.021457 = 0.101646}$$

$$SY_2 = \sqrt{\frac{0.013732}{9} + (0.107398)^2 (- 0.72931)^2 + (0.107398)^2(0.65699)^2 \ldots}$$

$$\ldots - 2(-0.009735) (- 0.72931) (0.65699)$$

$$= \sqrt{0.001526 + 0.006135 + 0.004979 - 0.009329 = 0.057541}$$

$$E_i = t*SY_i$$

$$E_1 = 2.447 * 0.101646 = 0.249$$

$$E_2 = 2.447 * 0.057541 = 0.141$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 2.133 - 0.249 = 1.884$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 2.133 + 0.249 = 2.382$$

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
0.000	2.079	2.133	0.249	1.884	2.381
0.693	2.303	2.202	0.141	2.062	2.343
1.099	2.197	2.182	0.125	2.058	2.307
1.386	2.079	2.112	0.131	1.982	2.243
1.609	1.946	1.999	0.134	1.865	2.134
1.792	1.792	1.838	0.131	1.707	1.968
1.946	1.792	1.608	0.125	1.483	1.732
2.079	1.099	1.262	0.141	1.122	1.403
2.197	0.693	0.643	0.249	0.394	0.891

Scatter Diagram and Plot of Beta Regression:

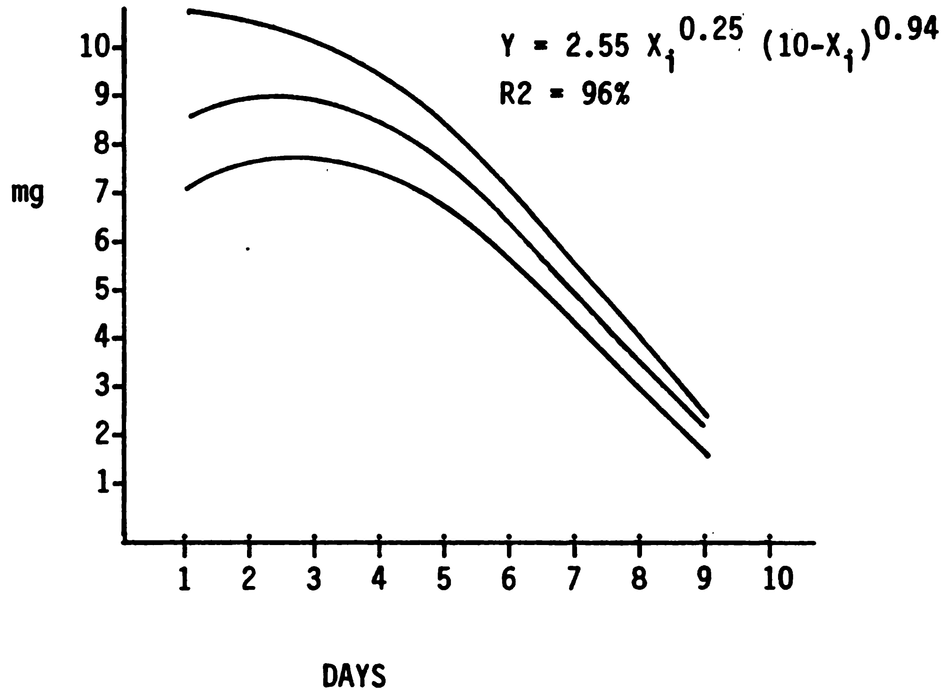

Figure 23. Beta Regression of concentration of drugs on days.

<u>Royleigh Regression Model</u>

$$Y_i = a\, X_i^{\,b}\, e^{cX2_i}$$

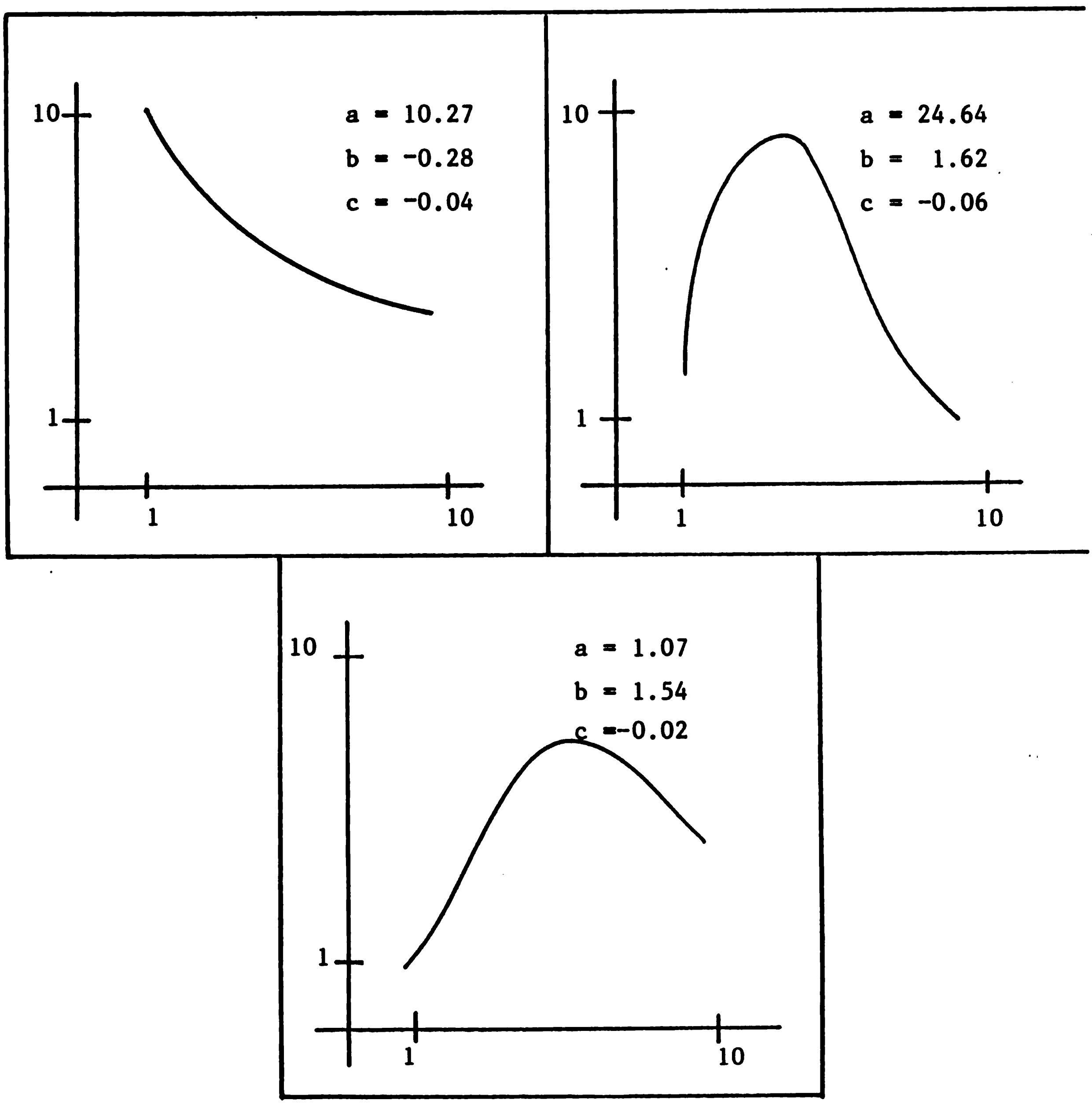

Figure 24. Spectrum of Royleigh Regression according to values of $\hat{a}$, $\hat{b}$ and $\hat{c}$.

EXAMPLE 10.

(Same data as presented in Example 1)

Time (days)	1	2	3	4	5	6	7	8	9
Concentration (mg %)	8	10	9	8	7	6	6	3	2

Steps to follow:

a) Fit a second order regression model to the sample data.

b) Estimate population parameters based on the sample.

c) Examine estimated plot, and scatter diagram and decide whether the royleigh regression model is the one that best describes the relationship between time and concentration.

d) Present results as an Analysis of Variance Table.

e) Display estimated $\hat{Y}_i$ values and confidence limits in tabular form.

Tabulation of Sum Squares and Cross Products

	var-X_i	var-Y_i	var-Z_i	X_i**2	X_i**3	X_i**4	X_i*Y_i	Z_i*Y_i
	0.0000	2.0794	1	0.0000	0.0000	1	0.00000	2.0794
	0.6931	2.3026	4	0.4804	2.7724	16	1.59593	9.2104
	1.0986	2.1972	9	1.2069	9.8874	81	2.41384	19.7748
	1.3863	2.0794	16	1.9218	22.1808	256	2.88267	33.2704
	1.6094	1.9459	25	2.5902	40.2350	625	3.13173	48.6475
	1.7918	1.7918	36	3.2105	64.5048	1296	3.21055	64.5048
	1.9459	1.7918	49	3.7865	95.3491	2401	3.48664	87.7892
	2.0794	1.0986	64	4.3239	133.0816	4096	2.28443	70.3104
	2.1972	0.6931	81	4.8277	177.9732	6561	1.52288	56.1411
SUM:	12.8017	15.9798	285	22.3479	545.9843	15333	20.52869	391.7370
MEAN:	1.42241	1.77553	31.66666					

Computation of Estimators:

$$\hat{b} = \frac{(SXY - SX*SY/n)(SX4 - SZ*SZ/n) - (SZY - SZ*SY/n)(SX3 - SX*SZ/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(20.52869 - 12.8017 * 15.9798/9)(15333 - 285 * 285/9) - \ldots}{(22.3479 - 12.8017*12.8017)(15333 - 285 * 285/9 -}$$

$$= \frac{\ldots(391.737 - 285 * 15.9798/9)\ (545.9843 - 12.8017*285/9)}{(545.9843- 12.8017*285/9)\ (545.9843 - 12.8017 * 285/9)}$$

$$= \frac{(-2.201154)(6308) - (- 114.29)(140.59714)}{(4.13862)(6308) - (140.5914)(140.59714)}$$

$$= \frac{2183.968}{6338.859}$$

$$= 0.344536$$

$$\hat{c} = \frac{(SX2 - SX*SX/n)(SZY - SZ*SY/n) - (SX3 - SX*SZ/n)(SXY - SX*SY/n)}{(SX2 - SX*SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)(SX3 - SX*SZ/n)}$$

$$= \frac{(22.3479 - 12.8017*12.8017/9)(391.737 - 285 * 15.9798/9) - \ldots}{(22.3479 - 12.8017*12.8017/9)(15333 - 285 * 285/9) -}$$

$$\frac{\ldots(545.9843 - 12.8017*285/9)(20.52869 - 12.8017*15.9798/9)}{(545.9843 - 12.8017*12.8017/9)(1545.9843 - 12.8017*285/9)}$$

$$= \frac{(4.13862)(-114.29) - (140.59714)(-2.201154)}{(4.13862)(6308) - (140.59714)(140.59714)}$$

$$= - \frac{163.52692}{6338.859}$$

$$= - 0.025797$$

$$\hat{a} = \hat{Y} - \hat{b}\bar{X} - \hat{C}\bar{Z}$$

$$= 1.77553 - (0.344536)(1.42241)-(-0.025797)(31.66666)$$

$$= 2.102363$$

Total S.S. = SY2 - SY * SY/n

= 30.6726 - 15.9798 * 15.9798/n

= 2.2999

Regression S.S

$$= \hat{b}*(SXY - SX*SY/n) + \hat{c}* (SZY - SZ*SY/n)$$

$$= (0.344536)(-2.201154) + (-0.025797)(-114.29)$$

$$= 2.19$$

Residual S.S. = Total S.S. - Regression S.S.

= 2.2999 - 2.19

= 0.1099

Tabulation of Royleigh Regression Analysis of Variance

Source of Variation	S.S.	D.F.	M.S.	F.
Regression............	2.1900	2	1.095	59.77
Residual	0.1099	6	0.01832	
Total	2.2999	8		

Computation of Reliability and Student's T test:

$$R2 = \frac{\text{Regression S.S.}}{\text{Total S.S.}} * 100$$

$$= \frac{2.19}{2.2999} * 100$$

= 95.22%

$$sb = \sqrt{rMS.* \frac{SX4 - SZ*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.0.01832 * \frac{6308}{6338.859}}$$

$$= 0.135021$$

$$sc = \sqrt{rMS.* \frac{SX2 - SX*SX/n}{(SX2 - SX * SX/n)(SX4 - SZ2*SZ2/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.01832 * \frac{4.13862}{6338.859}}$$

$$= 0.00345$$

$$(sbc) = \sqrt{rMS.* \frac{SX3 - SX*SZ/n}{(SX2 - SX * SX/n)(SX4 - SZ*SZ/n) - (SX3 - SX*SZ/n)^2}}$$

$$= \sqrt{0.01832 * - \frac{140.59714}{6338.859}}$$

$$= 0.020157$$

$$t = \frac{\hat{b}}{sb} = \frac{0.34536}{0.135021} = 2.55$$

$$t = \frac{\hat{c}}{sc} = - \frac{0.025795}{0.00345} = - 7.48$$

Computation of Y_i Estimates and Standard Errors of $\hat{Y}_i$:

$$\hat{Y}_i = \hat{a} + \hat{b} * (X_i) + \hat{c} * (Z_i)$$

$$\hat{Y}_1 = 2.102363 + (0.344536)\ (0) + (-0.025797)(1) = 2.077$$

$$\hat{Y}_2 = 2.102363 + (0.344536)(0.6931) + (-\ 0.025797)(4) = 2.238$$

$$SY_i = \sqrt{\frac{rMs}{n} + (sb)^2(X_i - \bar{X})^2 + (sc)^2\ (Z_i - \bar{Z})^2 - 2(sbc)^2\ (X_i - \bar{X})(Z_i - \bar{Z})}$$

$$SY_1 = \sqrt{\frac{0.01832}{9} + (0.135021)^2\ (-1.42241)^2 + (0.00345)^2\ (-30.66666)^2 \ldots}$$

$$\ldots - 2(0.020157)^2\ (-1.42241)(-30.66666)$$

$$= \sqrt{0.002036 + 0.036885 + 0.011191 - 0.035444 = 0.121112}$$

$$SY_2 = \sqrt{\frac{0.01832}{9} + (0.135021)^2\ (-\ 0.72931)^2 + (0.00345)^2\ (-27.66666)^2 \ldots}$$

$$\ldots - 2(-\ 0.020157)^2\ (-\ 0.72931)\ (-27.66666)$$

$$= \sqrt{0.002036 + 0.009697 + 0.009109 - 0.016395 = 0.066686}$$

$$E_i = t*SY_i$$

$$E_1 = 2.447 * 0.121112 = 0.296$$

$$E_2 = 2.447 * 0.066686 = 0.163$$

Computation of Confidence Limits:

$$\text{Lower c.l.} = \hat{Y}_1 - E_1$$

$$= 2.077 - 0.296 = 1.781$$

$$\text{Upper C.L.} = \hat{Y}_1 + E_1$$

$$= 2.077 + 0.296 = 2.373$$

Tabulation of Observed, Estimated and Confidence Limits

				Confidence Limits	
var-X	var-Y	Y-Hat	Error	Lower	Upper
0.000	2.097	2.077	0.297	1.780	2.373
0.693	2.303	2.238	0.164	2.074	2.402
1.099	2.197	2.249	0.157	2.092	2.406
1.386	2.079	2.167	0.165	2.003	2.332
1.609	1.946	2.012	0.159	1.853	2.171
1.792	1.792	1.791	0.143	1.648	1.934
1.946	1.792	1.509	0.139	1.369	1.648
2.079	1.099	1.168	0.175	0.992	1.343
2.197	0.693	0.770	0.257	0.513	1.026

Scatter Diagram and Plot of Royleigh Regression:

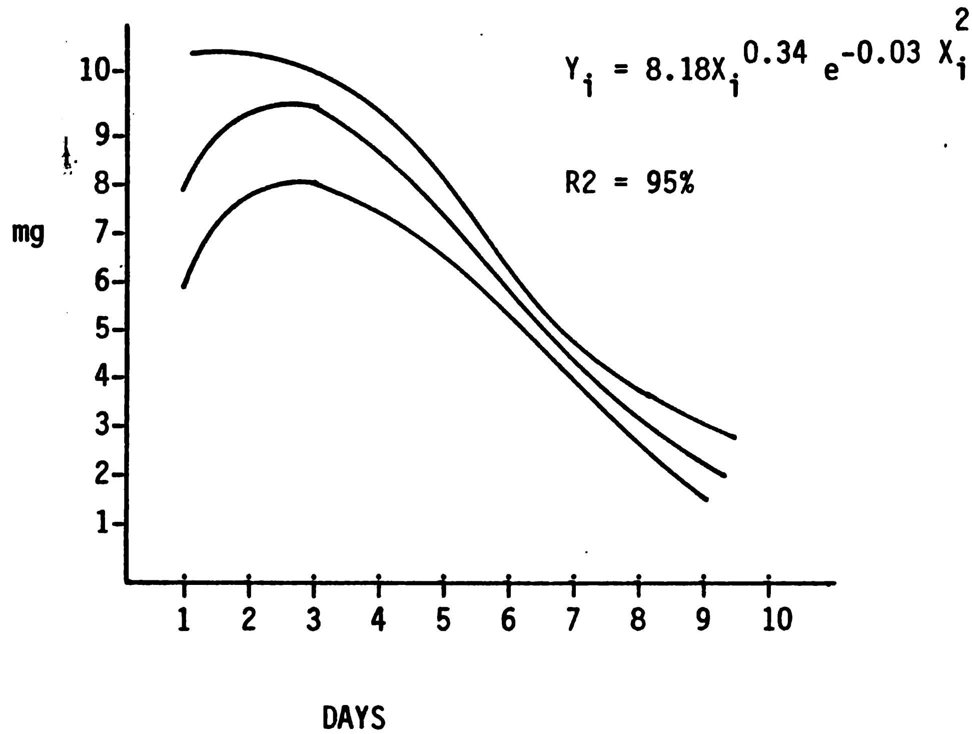

Figure 25. Royleigh Regression of concentration of drugs on days.

Appendix A: Computer Program for First Order Linear Regression Models.

In this appendix is listed the computer program written in Microsoft Basic Language for Wang Microcomputers.

```
1000 OPTION BASE 1
1010 REM ***************************************************
1020 REM * Program name:  First Order Regression Models    *
1030 REM * Filename:      "b:Regrelin.bas"                 *
1040 REM * Purpose:       Simple Regression Analysis       *
1050 REM *                based on 5 models.               *
1060 REM *                                                 *
1070 REM * Description of parameters:                      *
1080 REM *                X - Independent variable         *
1090 REM *                Y - Responce Variable            *
1100 REM *                n - Number of Data Points        *
1110 REM *                t - Value of Student's T         *
1120 REM *                                                 *
1130 REM * Remarks:       Enter t value with n - 2         *
1140 REM *                degrees of freedom in order      *
1150 REM *                to evaluate upper and lower      *
1160 REM *                confidence limits.               *
1170 REM ***************************************************
1180 CLS
1190 PRINT "***************************************************"
1200 PRINT "*      Linear Regression Analysis                *"
1210 PRINT "*      -------------------------                 *"
1220 PRINT "*                                                *"
1230 PRINT "*        Planning Unit, MANR.                    *"
1240 PRINT "*           Jan 10th, 1986.                      *"
1250 PRINT "***************************************************"
1260 PRINT "Enter Title of Experiment and Measurement Units"
1270 INPUT NA$
1280 INPUT "Enter Number of X-Y data points :";N
1290 PRINT "Enter Value of T taken from Statistical tables "
1300 INPUT "for n-2 degrees of freedom and 0.05 prob. level";T
1310 DIM X(40),Y(40),YH(40),YBY(40),XB(40),SSY(40),TSY(40)
1320 DIM XW(40),YW(40),LI(40),LE(40)
1330 FOR I = 1 TO N
1340 PRINT "No. : ";I
1350 INPUT "          Value of  X : ";XW(I)
1360 INPUT "          Value of  Y : ";YW(I)
1370 NEXT I
1380 M$(1)="Mean of var.... X = "
1390 M$(2)="Mean of var.... Y = "
1400 M$(3)="Reliability....R2 = "
1410 M$(4)="ST. ERROR.....SSb = "
1420 M$(5)="Student's.......T = "
1430 M$(6)="Constant........A = "
1440 M$(7)="Coeffic.........B = "
```

```
1450 R$(1)="1.LINEAR MODEL:     Y = A + B * X"
1460 R$(2)="2.SEMILOG MODEL:    Y = A + B * LN(X)"
1470 R$(3)="3.LOGARITH MODEL:   Y = A * X ** B"
1480 R$(4)="4.GEOMETRIC MODEL: Y = A * B ** X"
1490 R$(5)="5.RECIPROCAL MODEL: Y = A + B / X"
1500 R$(6)="Regression...."
1510 R$(7)="Residual......"
1520 R$(8)="Total........."
1530 CLS
1540 PRINT "****************************************"
1550 PRINT "*     1st ORDER MODELS MENU          *"
1560 PRINT "*     ---------------------          *"
1570 PRINT "*        1. Linear                   *"
1580 PRINT "*        2. Semilogarithm            *"
1590 PRINT "*        3. Logarithm                *"
1600 PRINT "*        4. Geometric                *"
1610 PRINT "*        5. Reciprocal               *"
1620 PRINT "*        6. Quit                     *"
1630 PRINT "****************************************"
1640 PRINT "Enter your selection   [Type in one number] "
1650 INPUT "Check Printer's paper and ..Press [RETURN]";IJK
1660 IF IJK < 1 OR IJK > 6 THEN 1670 ELSE 1690
1670 PRINT "ERROR: in Menu Selection......Try again"
1680 GOTO 1530
1690 ON IJK GOTO 1700,1770,1840,1920,2000,2070
1700 REM Linear Regression.............
1710 FOR I = 1 TO N
1720 X(I) = XW(I)
1730 Y(I) = YW(I)
1740 NEXT I
1750 GOSUB 2120
1760 GOTO 1530
1770 REM Semilogarithm Regression.......
1780 FOR I = 1 TO N
1790 X(I) = LOG(XW(I))
1800 Y(I) = YW(I)
1810 NEXT I
1820 GOSUB 2120
1830 GOTO 1530
1840 REM Logarithm Regression...........
1850 FOR I = 1 TO N
1860 X(I) = LOG(XW(I))
1870 Y(I) = LOG(YW(I))
1880 NEXT I
1890 GOSUB 2120
1900 GOSUB 2910
1910 GOTO 1530
1920 REM Geometric Regression...........
1930 FOR I = 1 TO N
1940 X(I) = XW(I)
1950 Y(I) = LOG(YW(I))
1960 NEXT I
```

```
1970 GOSUB 2120
1980 GOSUB 2910
1990 GOTO 1530
2000 REM Reciprocal Regression..........
2010 FOR I = 1 TO N
2020 X(I) = 1/(XN(I))
2030 Y(I) = YN(I)
2040 NEXT I
2050 GOSUB 2120
2060 GOTO 1530
2070 REM Quit...........................
2080 PRINT ">>>>>>>>>>>>>>>>"
2090 PRINT "........all done.......Type in [system] and press [RETURN]"
2100 PRINT ">>>>>>>>>>>>>>>>"
2110 END
2120 REM Sub Sum of Squares and Cross Products
2130 SX=0:SY=0:SXX=0:SXY=0:SYY=0
2140 FOR I = 1 TO N
2150 SX = SX + X(I)
2160 SY = SY + Y(I)
2170 SXX= SXX + X(I) * X(I)
2180 SXY= SXY + X(I) * Y(I)
2190 SYY= SYY + Y(I) * Y(I)
2200 NEXT I
2210 M(1)=SX / N
2220 M(2)=SY / N
2230 NU = SXY - SX * SY / N
2240 DE = SXX - SX * SX / N
2250 SCT= SYY - SY * SY / N
2260 M(7)=NU / DE
2270 M(6)=M(2)- M(7) * M(1)
2280 SCR= M(7) * NU
2290 SCE= SCT - SCR
2300 GLT= N - 1
2310 GLR= 1
2320 GLE= GLT - GLR
2330 CMR= SCR / GLR
2340 CME= SCE / GLE
2350 F  = CMR / CME
2360 M(3)=(SCR / SCT) *100!
2370 M(4)=SQR (CME/DE)
2380 M(5)=M(7) / M(4)
2390 FOR I = 1 TO N
2400 YH(I) = M(6)+M(7)*X(I)
2410 YBY(I)= YH(I) - Y(I)
2420 XB(I) = (X(I) -M(1))* (X(I) -M(1))
2430 SSY(I)= SQR(CME * (1/N + (XB(I)/DE)))
2440 TSY(I)= T * SSY(I)
2450 LI(I) = YH(I) - TSY(I)
2460 LE(I) = YH(I) + TSY(I)
2470 NEXT I
```

```
2480 LPRINT
2490 LPRINT TAB(20);CHR$(14);"PLANNING UNIT - MANR."
2500 LPRINT TAB(20);CHR$(14);"----------------------"
2510 LPRINT
2520 LPRINT TAB(6);CHR$(14);R$(IJK)
2530 LPRINT TAB(20);CHR$(20);" "
2540 LPRINT TAB(20);CHR$(27);"G";NA$
2550 LPRINT:LPRINT
2560 LPRINT TAB(20);"Table 1. Regression Analysis of Variance"
2570 LPRINT TAB(20);"         =============================="
2580 LPRINT CHR$(27);"H";" "
2590 L$="------------------------------------------------------------"
2600 LPRINT TAB(10);L$
2610 LPRINT TAB(10);   "Var. Source        S.S.      D.F.      M.S.      F."
2620 LPRINT TAB(10);L$
2630 LPRINT
2640 LPRINT TAB(10);R$(6);
2650 LPRINT USING "######.##";SCR;GLR;CMR;F
2660 LPRINT TAB(10);R$(7);
2670 LPRINT USING "######.##";SCE;GLE;CME
2680 LPRINT TAB(10);L$
2690 LPRINT TAB(10);R$(8);
2700 LPRINT USING "######.##";SCT;GLT
2710 LPRINT TAB(10);CHR$(27);"G"
2720 LPRINT TAB(20);"Table 2. Sample Statistics."
2730 LPRINT TAB(20);"         =================="
2740 LPRINT CHR$(27);"H";" "
2750 FOR I = 1 TO 7
2760 LPRINT TAB(10);M$(I);USING "########.###";M(I)
2770 NEXT I
2780 LPRINT TAB(10);CHR$(27);"G"
2790 LPRINT TAB(20);"Table 3. Observed and Expected  Values."
2800 LPRINT TAB(20);"         ============================="
2810 LPRINT CHR$(27);"H";" "
2820 LPRINT TAB(10);L$
2830 LPRINT TAB(49);"Confidence Limits"
2840 LPRINT TAB(10);   "    Var X     Var Y     Y Hat     Error     Lower     Upper"
2850 LPRINT TAB(10);L$
2860 FOR I = 1 TO N
2870 LPRINT TAB(10);USING "#####.###";X(I);Y(I);YH(I);TSY(I);LI(I);LE(I)
2880 NEXT I
2890 LPRINT TAB(10);L$
2900 RETURN
2910 REM sub antilog
2920 LPRINT TAB(10);".....Y - HATS .....(antilogs) FOLLOWS NEXT LINES:"
2930 LPRINT TAB(9);" ";
2940 FOR I = 1 TO N
2950 YH(I) = EXP(YH(I))
2960 LPRINT USING "####.#";YH(I);
2970 NEXT I
2980 LPRINT
2990 LPRINT TAB(10);L$
3000 LPRINT:LPRINT
3010 RETURN
```

PLANNING UNIT — MANR.

1. LINEAR MODEL: Y = A + B * X

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	46.82	1.00	46.82	34.84
Residual......	9.41	7.00	1.34	
Total.........	56.22	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	5.000
Mean of var.... Y =	6.556
Reliability....R2 =	83.271
ST. ERROR.....SSb =	0.150
Student's.......T =	-5.903
Constant........A =	10.972
Coeffic.........B =	-0.883

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Upper
1.000	8.000	10.089	1.685	8.404	11.774
2.000	10.000	9.206	1.401	7.805	10.606
3.000	9.000	8.322	1.156	7.166	9.478
4.000	8.000	7.439	0.980	6.459	8.419
5.000	7.000	6.556	0.914	5.642	7.469
6.000	6.000	5.672	0.980	4.692	6.652
7.000	6.000	4.789	1.156	3.633	5.945
8.000	3.000	3.906	1.401	2.505	5.306
9.000	2.000	3.022	1.685	1.337	4.707

PLANNING UNIT — MANR.

2. SEMILOG MODEL: Y = A + B * LN(X)

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	33.01	1.00	33.01	9.96
Residual......	23.21	7.00	3.32	
Total.........	56.22	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	1.422
Mean of var.... Y =	6.556
Reliability....R2 =	58.719
ST. ERROR.....SSb =	0.895
Student's.......T =	-3.155
Constant........A =	10.573
Coeffic.........B =	-2.824

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Upper
0.000	8.000	10.573	3.336	7.237	13.909
0.693	10.000	8.615	2.108	6.507	10.723
1.099	9.000	7.470	1.591	5.879	9.061
1.386	8.000	6.658	1.438	5.220	8.095
1.609	7.000	6.027	1.489	4.538	7.516
1.792	6.000	5.512	1.635	3.878	7.147
1.946	6.000	5.077	1.813	3.264	6.890
2.079	3.000	4.700	1.999	2.701	6.699
2.197	2.000	4.367	2.180	2.188	6.547

PLANNING UNIT — MANR.

3. LOGARITH MODEL: Y = A * X ** B

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	1.17	1.00	1.17	7.26
Residual......	1.13	7.00	0.16	
Total.........	2.30	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	1.422
Mean of var.... Y =	1.776
Reliability....R2 =	50.909
ST. ERROR.....SSb =	0.197
Student's.......T =	-2.694
Constant........A =	2.532
Coeffic.........B =	-0.532

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
0.000	2.079	2.532	0.736	1.796	3.268
0.693	2.303	2.163	0.465	1.699	2.628
1.099	2.197	1.948	0.351	1.597	2.299
1.386	2.079	1.795	0.317	1.478	2.112
1.609	1.946	1.676	0.328	1.348	2.004
1.792	1.792	1.579	0.360	1.219	1.940
1.946	1.792	1.497	0.400	1.097	1.897
2.079	1.099	1.426	0.441	0.985	1.867
2.197	0.693	1.363	0.481	0.883	1.844

.....Y - HATS .....(antilogs) FOLLOWS NEXT LINES:

12.6 8.7 7.0 6.0 5.3 4.9 4.5 4.2 3.9

PLANNING UNIT – MANR.

4. GEOMETRIC MODEL: Y = A * B ** X

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	1.75	1.00	1.75	22.45
Residual......	0.55	7.00	0.08	
Total.........	2.30	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	5.000
Mean of var.... Y =	1.776
Reliability....R2 =	76.229
ST. ERROR.....SSb =	0.036
Student's.......T =	-4.738
Constant........A =	2.630
Coeffic.........B =	-0.171

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
1.000	2.079	2.459	0.406	2.053	2.865
2.000	2.303	2.288	0.338	1.951	2.626
3.000	2.197	2.117	0.279	1.839	2.396
4.000	2.079	1.946	0.236	1.710	2.183
5.000	1.946	1.776	0.220	1.555	1.996
6.000	1.792	1.605	0.236	1.368	1.841
7.000	1.792	1.434	0.279	1.155	1.712
8.000	1.099	1.263	0.338	0.925	1.600
9.000	0.693	1.092	0.406	0.686	1.498

.....Y - HATS .....(antilogs) FOLLOWS NEXT LINES:

11.7 9.9 8.3 7.0 5.9 5.0 4.2 3.5 3.0

PLANNING UNIT – MANR.

5. RECIPROCAL MODEL: Y = A + B / X

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	16.83	1.00	16.83	2.99
Residual......	39.39	7.00	5.63	
Total.........	56.22	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	0.314
Mean of var.... Y =	6.556
Reliability....R2 =	29.936
ST. ERROR.....SSb =	2.941
Student's.......T =	1.729
Constant........A =	4.957
Coeffic.........B =	5.086

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
1.000	8.000	10.043	5.123	4.920	15.166
0.500	10.000	7.500	2.273	5.227	9.773
0.333	9.000	6.652	1.875	4.777	8.527
0.250	8.000	6.228	1.923	4.305	8.151
0.200	7.000	5.974	2.032	3.942	8.006
0.167	6.000	5.804	2.134	3.671	7.938
0.143	6.000	5.683	2.218	3.465	7.901
0.125	3.000	5.593	2.287	3.305	7.880
0.111	2.000	5.522	2.344	3.178	7.866

Appendix B: Computer Program for Second Order Linear Regression Models

In this appendix is listed the computer program written in Microsoft Basic Language for Wang Microcomputers.

```
1000 OPTION BASE 1
1010 REM ****************************************************
1020 REM * Program name:  Second Order Regression Models  *
1030 REM * Filename:      "b:Regrecur.bas"                *
1040 REM * Purpose:       Quadratic  Regression Analysis  *
1050 REM *                based on 5 models.              *
1060 REM * Description of parameters:                     *
1070 REM *                X - Independent variable        *
1080 REM *                Y - Responce Variable           *
1090 REM *                Z - Quadratic form              *
1100 REM *                n - Number of Data Points       *
1110 REM *                t - Value of Student's T        *
1120 REM * Remarks:       Enter t value with n - 3        *
1130 REM *                degrees of freedom in order     *
1140 REM *                to evaluate upper and lower     *
1150 REM *                confidence limits.              *
1160 REM ****************************************************
1170 CLS
1180 PRINT "**************************************************"
1190 PRINT "*     Linear Regression Analysis                 *"
1200 PRINT "*     --------------------------                 *"
1210 PRINT "*                                                *"
1220 PRINT "*       Planning Unit, MANR.                     *"
1230 PRINT "*          Jan 10th, 1986.                       *"
1240 PRINT "**************************************************"
1250 PRINT "Enter Title of Experiment and Measurement Units"
1260 INPUT NA$
1270 INPUT "Enter number of X and Y data points :";N
1280 PRINT "Enter Value of T taken from Statistical tables "
1290 INPUT "for n-3 degrees of freedom and 0.005 prob. level";T
1300 DIM X(40),Y(40),YH(40),SSY(40),TSY(40),Z(40),M$(12),M(12)
1310 DIM XW(40),YW(40),LI(40),LE(40),XB1(40),XB2(40),XB3(40),XBY(40)
1320 FOR I = 1 TO N
1330 PRINT "No. : ";I
1340 INPUT "          Value of  X : ";XW(I)
1350 INPUT "          Value of  Y : ";YW(I)
1360 NEXT I
1370 M$(1)="Mean of var.... X = "
1380 M$(2)="Mean of var.... Y = "
1390 M$(3)="Mean of Trans.. Z = "
1400 M$(4)="Reliability....R2 = "
1410 M$(5)="ST. ERROR.....SSb = "
1420 M$(6)="ST. ERROR.....SSc = "
1430 M$(7)="Student's......Tb = "
1440 M$(8)="Student's......Tc = "
1450 M$(9)="Constant........A = "
1460 M$(10)="Coeffic.........B = "
1470 M$(11)="Coeffic.........C = "
1480 M$(12)="COVARIANCE.....bc = "
```

```
1490 R$(1)="1.QUADRATIC:Y= A + B * X + C * X ** 2"
1500 R$(2)="2.SQ.ROOT: Y = A + B * X + C * SQ(X)"
1510 R$(3)="3.GAMMA: Y = A * EXP(B * X) * X ** C)"
1520 R$(4)="4.BETA: Y= A * X ** B * (K - X) ** C"
1530 R$(5)="5.ROYLEIGH:Y= A * X**B *EXP(C * X**2)"
1540 R$(6)="Regression...."
1550 R$(7)="Error........."
1560 R$(8)="Total........."
1570 CLS
1580 PRINT "*****************************************"
1590 PRINT "*        LINEAL MODELS MENU             *"
1600 PRINT "*        ------------------             *"
1610 PRINT "*        1. Quadratic                   *"
1620 PRINT "*        2. Square Root                 *"
1630 PRINT "*        3. Gamma                       *"
1640 PRINT "*        4. Beta                        *"
1650 PRINT "*        5. Royleigh                    *"
1660 PRINT "*        6. Quit                        *"
1670 PRINT "*****************************************"
1680 PRINT "Enter your selection  [Type in one number] "
1690 INPUT "Check Printer's paper and ..Press [RETURN]";IJK
1700 IF IJK < 1 OR IJK > 6 THEN 1710 ELSE 1730
1710 PRINT "ERROR: in Menu Selection......Try again"
1720 GOTO 1570
1730 ON IJK GOTO 1740,1820,1900,1990,2070,2160
1740 REM Quadratic Regression...........
1750 FOR I = 1 TO N
1760 X(I) = XW(I)
1770 Y(I) = YW(I)
1780 Z(I) = XW(I) * XW(I)
1790 NEXT I
1800 GOSUB 2210
1810 GOTO 1570
1820 REM Square Root Regression..........
1830 FOR I = 1 TO N
1840 X(I) = XW(I)
1850 Y(I) = YW(I)
1860 Z(I) = SQR(XW(I))
1870 NEXT I
1880 GOSUB 2210
1890 GOTO 1570
1900 REM Gamma Regression...............
1910 FOR I = 1 TO N
1920 Z(I) = LOG(XW(I))
1930 Y(I) = LOG(YW(I))
1940 X(I) = XW(I)
1950 NEXT I
1960 GOSUB 2210
1970 GOSUD 3180
1980 GOTO 1570
```

```
1990 REM Beta Regression................
1995 INPUT "Enter Value for K Constant";KKK
2000 FOR I = 1 TO N
2010 X(I) = LOG(XW(I))
2020 Y(I) = LOG(YW(I))
2030 Z(I) = LOG (KKK - XW(I))
2040 NEXT I
2050 GOSUB 2210
2060 GOTO 1570
2070 REM Royleigh Regression............
2080 FOR I = 1 TO N
2090 X(I) = LOG(XW(I))
2100 Y(I) = LOG(YW(I))
2110 Z(I) = XW(I) * XW(I)
2120 NEXT I
2130 GOSUB 2210
2140 GOSUB 3180
2150 GOTO 1570
2160 REM Quit............................
2170 PRINT ">>>>>>>>>>>>>>>>"
2180 PRINT "........all done.......Type in [system] and press [RETURN]"
2190 PRINT ">>>>>>>>>>>>>>>>"
2200 END
2210 REM Sub Sum of Squares and Cross Products
2220 SX=0:SY=0:SYY=0:SXY=0:SZ=0:SZY=0:SX2=0:SX3=0:SX4=0
2230 FOR I = 1 TO N
2240 SX = SX + X(I)
2250 SX2 = SX2 + X(I) * X(I)
2260 SX3 = SX3 + X(I) * Z(I)
2270 SX4 = SX4 + Z(I) * Z(I)
2280 SY = SY + Y(I)
2290 SYY = SYY + Y(I) * Y(I)
2300 SXY = SXY + X(I) * Y(I)
2310 SZY = SZY + Z(I) * Y(I)
2320 SZ = SZ + Z(I)
2330 NEXT I
2340 M(1) = SX / N
2350 M(2) = SY / N
2360 M(3) = SZ / N
2370 NU1 = (SXY - SX * SY / N) * (SX4 - SZ * SZ / N)
2380 NU2 = (SZY - SZ * SY / N) * (SX3 - SX * SZ / N)
2390 NU = NU1 - NU2
2400 DE1 = (SX4 - SZ * SZ / N) * (SX2 - SX * SX / N)
2410 DE2 = (SX3 - SX * SZ / N) * (SX3 - SX * SZ / N)
2420 DE = DE1 - DE2
2430 SCT = SYY - SY * SY / N
2440 M(10) = NU / DE
2450 NU1 = (SX2 - SX * SX / N) * (SZY - SY * SZ / N)
2460 NU2 = (SX3 - SZ * SX / N) * (SXY - SX * SY / N)
2470 NU = NU1 - NU2
2480 M(11) = NU / DE
```

```
2490 M(9) = M(2) - M(10) * M(1) - M(11) * M(3)
2500 SCR= (M(10)*(SXY-SX*SY/N)) + (M(11)*(SZY-SZ*SY/N))
2510 SCE= SCT - SCR
2520 GLT= N - 1
2530 GLR= 2
2540 GLE= GLT - GLR
2550 CMR= SCR / GLR
2560 CME= SCE / GLE
2570 F  = CMR / CME
2580 M(4) = (SCR/SCT)*100
2590 M(5) =SQR(CME*(SX4 - SZ * SZ / N)/DE)
2600 M(6) =SQR(CME*(SX2 - SX * SX / N)/DE)
2610 M(12)= CME*(SX3 - SZ * SX / N)/DE
2620 M(7) = M(10) / M(5)
2630 M(8) = M(11) / M(6)
2640 FOR I = 1 TO N
2650 YH(I) = M(9) + M(10)*X(I) + M(11) * Z(I)
2660 YBY(I) = YH(I) - Y(I)
2670 XB1(I) = (X(I) - M(1))
2680 XB2(I) = (Z(I) - M(3))
2690 XB3(I) = (X(I) - M(1)) * (Z(I) - M(3))
2700 SSY(I) = SQR( CME/N + M(5)*M(5)*XB1(I) * XB1(I) + M(6)*M(6)*XB2(I)
     * XB2(I) - 2 * M(12)*XB1(I)*XB2(I))
2710 TSY(I) = T * SSY(I)
2720 LI(I) = YH(I) - TSY(I)
2730 LE(I) = YH(I) + TSY(I)
2740 NEXT I
2750 LPRINT
2760 LPRINT TAB(20);CHR$(14);"PLANNING UNIT - MANR."
2770 LPRINT TAB(20);CHR$(14);"---------------------"
2780 LPRINT
2790 LPRINT TAB(6);CHR$(14);R$(IJK)
2800 LPRINT TAB(20);CHR$(20);" "
2810 LPRINT TAB(20);CHR$(27);"G";MA$
2820 LPRINT:LPRINT
2830 LPRINT TAB(20);"Table 1. Regression Analysis of Variance"
2840 LPRINT TAB(20);"          =============================="
2850 LPRINT CHR$(27);"H";" "
2860 L$="--------------------------------------------------------"
2870 LPRINT TAB(10);L$
2880 LPRINT TAB(10);    "Var. Source       S.S.     D.F.     M.S.     F."
2890 LPRINT TAB(10);L$
2900 LPRINT
2910 LPRINT TAB(10);R$(6);
2920 LPRINT USING "######.##";SCR;GLR;CMR;F
2930 LPRINT TAB(10);R$(7);
2940 LPRINT USING "######.##";SCE;GLE;CME
2950 LPRINT TAB(10);L$
2960 LPRINT TAB(10);R$(8);
2970 LPRINT USING "######.##";SCT;GLT
```

```
2980 LPRINT TAB(10);CHR$(27);"G"
2990 LPRINT TAB(20);"Table 2. Sample Statistics."
3000 LPRINT TAB(20);"        =================="
3010 LPRINT CHR$(27);"H";" "
3020 FOR I = 1 TO 12
3030 LPRINT TAB(10);M$(I);USING "########.###";M(I)
3040 NEXT I
3050 LPRINT TAB(10);CHR$(27);"G"
3060 LPRINT TAB(20);"Table 3. Observed and Expected  Values."
3070 LPRINT TAB(20);"        ============================"
3080 LPRINT CHR$(27);"H";" "
3090 LPRINT TAB(10);L$
3100 LPRINT TAB(49);"Confidence Limits"
3110 LPRINT TAB(10);    "    Var X    Var Y    Y Hat    Error    Lower    Upper"
3120 LPRINT TAB(10);L$
3130 FOR I = 1 TO N
3140 LPRINT TAB(10);USING "#####.###";X(I);Y(I);YH(I);TSY(I);LI(I);LE(I)
3150 NEXT I
3160 LPRINT TAB(10);L$
3170 RETURN
3180 REM sub antilog
3190 LPRINT TAB(10);".....Y - HATS (antilogs) FOLLOWS NEXT LINE :"
3200 LPRINT TAB(10);" ";
3210 FOR I = 1 TO N
3220 YH(I) = EXP(YH(I))
3230 LPRINT USING "####.#";YH(I);
3240 NEXT I
3250 LPRINT
3260 LPRINT TAB(10);L$
3270 LPRINT:LPRINT
3280 RETURN
```

PLANNING UNIT — MANR.

1. QUADRATIC: Y= A + B * X + C * X ** 2

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	52.64	2.00	26.32	44.02
Error.........	3.59	6.00	0.60	
Total.........	56.22	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	5.000
Mean of var.... Y =	6.556
Mean of Trans.. Z =	31.667
Reliability....R2 =	93.620
ST. ERROR.....SSb =	0.452
ST. ERROR.....SSc =	0.044
Student's......Tb =	1.087
Student's......Tc =	-3.120
Constant........A =	8.452
Coeffic.........B =	0.491
Coeffic.........C =	-0.137
COVARIANCE.....bc =	0.019

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
1.000	8.000	8.806	1.538	7.268	10.344
2.000	10.000	8.885	0.999	7.886	9.884
3.000	9.000	8.689	0.848	7.841	9.537
4.000	8.000	8.218	0.911	7.306	9.129
5.000	7.000	7.472	0.956	6.516	8.428
6.000	6.000	6.451	0.911	5.540	7.362
7.000	6.000	5.155	0.848	4.307	6.003
8.000	3.000	3.585	0.999	2.586	4.584
9.000	2.000	1,739	1.538	0.202	3.277

PLANNING UNIT — MANR.

2. SQ. ROOT: Y = A + B * X + C * SQ(X)

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	53.61	2.00	26.81	61.68
Error.........	2.61	6.00	0.43	
Total.........	56.22	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	5.000
Mean of var.... Y =	6.556
Mean of Trans.. Z =	2.145
Reliability....R2 =	95.362
ST. ERROR.....SSb =	0.597
ST. ERROR.....SSc =	2.442
Student's......Tb =	-5.394
Student's......Tc =	3.955
Constant........A =	1.940
Coeffic.........B =	-3.221
Coeffic.........C =	9.658
COVARIANCE.....bc =	1.443

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
1.000	8.000	8.378	1.450	6.928	9.828
2.000	10.000	9.158	0.825	8.333	9.983
3.000	9.000	9.007	0.801	8.206	9.809
4.000	8.000	8.375	0.817	7.557	9.192
5.000	7.000	7.434	0.765	6.669	8.199
6.000	6.000	6.275	0.687	5.588	6.961
7.000	6.000	4.950	0.687	4.262	5.637
8.000	3.000	3.494	0.863	2.631	4.356
9.000	2.000	1.930	1.200	0.730	3.130

PLANNING UNIT — MANR.

3. GAMMA: Y = A * EXP(B * X) * X ** C)

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	2.13	2.00	1.07	37.70
Error.........	0.17	6.00	0.03	
Total.........	2.30	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	5.000
Mean of var.... Y =	1.776
Mean of Trans.. Z =	1.422
Reliability....R2 =	92.629
ST. ERROR.....SSb =	0.073
ST. ERROR.....SSc =	0.278
Student's......Tb =	-5.827
Student's......Tc =	3.654
Constant........A =	2.459
Coeffic.........B =	-0.426
Coeffic.........C =	1.017
COVARIANCE.....bc =	0.019

Table 3. Observed and Expected Values.

				Confidence Limits	
Var X	Var Y	Y Hat	Error	Lower	Upper
1.000	2.079	2.033	0.381	1.652	2.414
2.000	2.303	2.312	0.211	2.101	2.523
3.000	2.197	2.298	0.212	2.087	2.510
4.000	2.079	2.165	0.207	1.957	2.372
5.000	1.946	1.966	0.187	1.779	2.153
6.000	1.792	1.725	0.168	1.557	1.893
7.000	1.792	1.456	0.174	1.282	1.630
8.000	1.099	1.166	0.220	0.946	1.386
9.000	0.693	0.860	0.297	0.563	1.156

.....Y - HATS (antilogs) FOLLOWS NEXT LINE :

7.6   10.1   10.0   8.7   7.1   5.6   4.3   3.2   2.4

PLANNING UNIT – MANR.

4. BETA: Y= A * X ** B * (K – X) ** C

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	2.22	2.00	1.11	80.80
Error.........	0.08	6.00	0.01	
Total.........	2.30	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	1.422
Mean of var.... Y =	1.776
Mean of Trans.. Z =	1.422
Reliability....R2 =	96.420
ST. ERROR.....SSb =	0.107
ST. ERROR.....SSc =	0.107
Student's......Tb =	2.418
Student's......Tc =	8.733
Constant........A =	0.072
Coeffic.........B =	0.260
Coeffic.........C =	0.938
COVARIANCE.....bc =	-0.010

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Confidence Limits Upper
0.000	2.079	2.133	0.249	1.884	2.381
0.693	2.303	2.202	0.141	2.062	2.343
1.099	2.197	2.182	0.125	2.058	2.307
1.386	2.079	2.112	0.131	1.982	2.243
1.609	1.946	1.999	0.134	1.865	2.134
1.792	1.792	1.838	0.131	1.707	1.968
1.946	1.792	1.608	0.125	1.483	1.732
2.079	1.099	1.262	0.141	1.122	1.403
2.197	0.693	0.643	0.249	0.394	0.891

PLANNING UNIT - MANR.

5. ROYLEIGH: Y= A * X**B *EXP(C * X**2)

Concentration of NN in Blood after T days. Jan-10-1986.

Table 1. Regression Analysis of Variance

Var. Source	S.S.	D.F.	M.S.	F.
Regression....	2.19	2.00	1.10	59.91
Error.........	0.11	6.00	0.02	
Total.........	2.30	8.00		

Table 2. Sample Statistics.

Mean of var.... X =	1.422
Mean of var.... Y =	1.776
Mean of Trans.. Z =	31.667
Reliability....R2 =	95.232
ST. ERROR.....SSb =	0.135
ST. ERROR.....SSc =	0.003
Student's......Tb =	2.554
Student's......Tc =	-7.468
Constant........A =	2.102
Coeffic.........B =	0.344
Coeffic.........C =	-0.026
COVARIANCE.....bc =	0.000

Table 3. Observed and Expected Values.

Var X	Var Y	Y Hat	Error	Confidence Limits Lower	Upper
0.000	2.079	2.077	0.297	1.780	2.373
0.693	2.303	2.238	0.164	2.074	2.402
1.099	2.197	2.249	0.157	2.092	2.406
1.386	2.079	2.167	0.165	2.003	2.332
1.609	1.946	2.012	0.159	1.853	2.171
1.792	1.792	1.791	0.143	1.648	1.934
1.946	1.792	1.509	0.139	1.369	1.648
2.079	1.099	1.168	0.175	0.992	1.343
2.197	0.693	0.770	0.257	0.513	1.026

.....Y - HATS (antilogs) FOLLOWS NEXT LINE :

8.0 9.4 9.5 8.7 7.5 6.0 4.5 3.2 2.2

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